Trading the randomness  
- Designing an optimal trading strategy  
under a drifted random walk price model

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Math 20 Project Paper
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Abstract:
In this paper the author intends to explore the application of probability mathematics in trading financial markets. To achieve that, the author will build a drifted random walk model to simulate the price movements of stocks, and then based on a set of assumptions about transaction cost and trading advantage, establish a trading system. The author then will examine the expected payoff from different trading strategies under this system and conclude an optimal trading strategy in this model. The appendix will then provide a Matlab program to simulate this model and test all conclusions in this paper.

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Part I. Background and introduction

Modeling stock price with some kind of random walk process is not a new idea. It was most famously noted by Princeton Economist Burton Malkiel’s book *A Random Walk Down Wall Street*¹, in which Malkiel argued that movements of stock price behave randomly and therefore the game of trading stocks is essentially a martingale². However, some other economists and many market participants believe otherwise. They argue that stock price does not behave completely like a random walk, and have suggested other price models. One price model developed by Andrew W. Lo and Archie Craig MacKinlay is the basis of my price model in this paper. It can be expressed as:

\[ X_t = \mu + X_{t-1} + \varepsilon_t \]

where

- \( X_t \) is the price of the stock at time \( t \)
- \( \mu \) is the trend force drift
- \( \varepsilon_t \) is the random disturbance term³.

To understand the above formula, one can think of stock price movement in a trended market. Each discrete movement is random (represented by \( \varepsilon_t \)), but there is also a trend force (represented by \( \mu \)) that drives the price towards certain direction.

(Chart 1.1)

For example, as the chart (Chart 1.1) above shows, each price movement is random (as there are ups and downs in a random sequence); however, as the blue arrow shows, overall there is a direction (or trend force) that price oscillates around. This trend direction could be driven by fundamental factors (e.g. in the example of a stock, the performance and quality of the company) or momentum factors (e.g. if many people are buying the same stock, the stock price is driven up, which in turn attracts

more buyers and drives price further up – a classic explanation of financial bubbles). Therefore, a skilled trader who understands those factors well could profit by guessing correctly this trend direction, in other words, predicting the correct \( \mu \).

**Part II. Introducing the variables and the assumptions**

To start with, I will define the variables in this model mathematically.

1) **Stock price \( X_t \)**
The variable \( X_t \) represents the stock price at time \( t \). It is the exact price at which the trader can buy or sell at. Again, in this model: \( X_t = \mu + X_{t-1} + \varepsilon_t \). As in this model we care only about the relative price movements, we could set \( X_0 = 0 \). Hence, we can derive that \( X_\Gamma = X_0 + \sum_{i} \mu_i + \sum_{i} \varepsilon_i = \sum_{i} \mu_i + \sum_{i} \varepsilon_i \). We will further adjust the formula for this variable below in this model to incorporate some other elements.

2) **The time \( t \)**
In this model time \( t \) is a discrete integer from 0 onwards, and therefore, the price movements is also a discrete process.

3) **The drift \( \mu \)**
In this model the drift \( \mu \) will be of a constant size \( \mu_0 \) representing the part of price movements due to the trend force in a unit time. As shown in Chart 1.1, one could easily derive the relationship between \( \mu_0 \) and the trend line angle \( \Theta \): \( \mu_0 = \tan \theta \).
However, the direction of the trend could go either way with equal probability. More details on how to model the sign of \( \mu \) will be discussed below.

4) **The random disturbance term \( \varepsilon_t \)**
As said, in this paper, the stock price will be modeled with a simple random walk with a drift, so the random disturbance term \( \varepsilon_t \) will behave as a simple random walk variable with a constant step size \( \varepsilon_0 \), so its probability distribution function is:

\[
P(\varepsilon_t) = \begin{cases} 
\frac{1}{2} & \varepsilon_t = \varepsilon_0 \\
\frac{1}{2} & \varepsilon_t = -\varepsilon_0 
\end{cases}
\]

Besides parameters in the original price model \((X_t, t, \mu, \varepsilon)\), I will also introduce several new variables into this price model below.

5) **The trend length \( \Gamma \)**
A trend in financial markets does not last forever. It dies out after certain time as the fundamental factors behind it are fully reflected in the price or as the momentum ends. Hence, \( \Gamma \) is used to represent how long (in the same unit of \( t \)) the trend lasts.
6) Trader’s edge \( \alpha \)
As the paper discussed above, if the price trends towards a certain direction (drift \( \mu \neq 0 \)), then a skilled trader might be able to gain an edge in trading by having a good guess about the direction of the drift. In other words, if the trader can guess correctly whether \( \mu \) is positive or negative for more than half of the time, then she might be able to develop a system that provides a positive expected return. In this model, we will assign a value \( \alpha \) to represent this trader’s edge. The trader will guess correctly \( \mu \) for a probability of \( 0.5 + \alpha \); in the other scenario with a probability of \( 0.5 - \alpha \), I assume that the trader still guess the correct absolute value of \( \mu \) but with the opposite sign (hence \(-\mu\)).

It is easy to see that, by symmetry, if \( \mu \) is positive and price is trending up, but the trader guesses wrong and decides to short - sell (essentially betting the price goes down), it is equivalent to the trader buys when the price is trending down (with a negative \( \mu \)). Therefore, we could conveniently convert all scenarios into one that the trader always buys when entering a position, but with \( \mu \) being positive when the trader guesses correctly and \( \mu \) being negative when the trader guesses wrong. Therefore, given a fixed size of drift \( \mu_0 \), the drift \( \mu \) in this case could be modeled with a probability distribution function:

\[
P(\mu) = \begin{cases} 
\frac{1}{2} + \alpha & \mu = \mu_0 \\
\frac{1}{2} - \alpha & \mu = -\mu_0 
\end{cases}
\]

7) Profit Limit \( L \) and Stop Loss \( S \)
To protect their capital, traders often set a Stop Loss (\( S \)) that when price goes against their plan and the loss exceeds this point, they will exit their position with a loss. As stated above, in this model we converted all scenario into one that trader buys when entering a position. Hence when \( X_t = X_{\text{enter}} - S \), the trader takes in a loss of \( S \) and exits the position.

While a Profit Limit (\( T \)) is exactly the opposite, it defines a gain that the trader is satisfied with and beyond which is unwilling to take further risk. Hence, in this model it represents that when \( X_t = X_{\text{enter}} + T \), the trader takes in a gain of \( T \) and exits the position.

Now it is important to note that, if the trader enters the position at \( X_{\text{enter}} \) and exits at \( X_{\text{exit}} \), it is essentially the same as entering at 0 and exits at \( X_{\text{exit}} - X_{\text{enter}} \).

\[
X_t = \mu + X_{t-1} + \varepsilon_t, \quad \text{so} \quad X_{\text{exit}} - X_{\text{enter}} = \mu(\text{exit} - \text{enter} - 1) + \sum_{t=\text{enter}+1}^{\text{exit}} \varepsilon_t.
\]

As \( \varepsilon_t \) are all independent and have the same distribution function, \( E(\sum_{t=\text{enter}+1}^{\text{exit}} \varepsilon_t) \) is no different from \( E(\sum_{t=0}^{\text{exit} - \text{enter} - 1} \varepsilon_t) \), so \( E(X_{\text{exit}} - X_{\text{enter}}) = \mu(\text{exit} - \text{enter} - 1) + E(\sum_{t=0}^{\text{exit} - \text{enter} - 1} \varepsilon_t) \). In other words, we could reset \( X_{\text{enter}} = X_0 = 0 \) and \( X_{\text{exit}} = X_{\text{exit-enter-1}} \) (of course with new random values for \( X_t \)) and maintain \( E(X_{\text{exit}} - X_{\text{enter}}) \). Hence, in this model, we would simply reset \( X_t \) to 0 after entering/re-entering a trade, and exit the trade when \( X_t \leq S \) (for a loss) or \( X_t \geq L \) (for a gain). Hence, we can get a new formula for \( X_t \):
\[ X_t = \sum_{t=enter}^{i} \mu_i + \sum_{t=enter}^{i} \epsilon_i \]

where \( t\)-enter is the enter time of the current trade.

8) Transaction cost \( C \)
As trading stocks involve a transaction cost (or often called a “commission fee”), this variable \( C \) denotes a fixed transaction cost for each trade.

9) Outcome of the trade \( O_i \)
The Outcome of the \( i \)th trade is denoted as \( O_i \). This could be a gain or loss. As introduced above, if the price moves over the Profit Limit \( L \) or the Stop Loss \( S \), the trader will take a gain of \( L \) or a loss of \( S \) respectively. Another case that the trader will exit the trader and take a gain/loss in this model is when the trend arrives to an end \((t=\Gamma)\), the trader will have to exit her position at \( X_\Gamma \) in order to prevent the risk of a sharp reversal (a big price change in the unwanted direction). These described rules along with the transaction cost \( C \) gives us:

\[
O_i = \begin{cases} 
L - C & X_t \geq L \\
-S - C & X_t \leq S \\
X_\Gamma - C & t = \Gamma 
\end{cases}
\]

10) Total profit of the strategy \( P \)
This variable \( P \) accounts for the sum of gains and losses of all trades during the length of the trade. Its value is the best measurement of how successful a trading strategy is. It is given by:

\[ P = O_1 + O_2 + O_3 + ... + O_n \]

\( n \) is the total number of trades done within time \( \Gamma \).

Part III. A simple case, no drift (\( \mu_0 = 0 \))

We start with a special case of the model: \( \mu_0 = 0 \), then \( \mu = 0 \) and \( X_t = X_{t-1} + \epsilon_t \). In this case, stock price is indeed a random walk. This situation happens when there is a flat market and price only oscillates horizontally and does not seem to move consistently towards either direction (as in Chart 3.1).
In this case, it is clear that the trader’s edge $\alpha$ becomes irrelevant. Under this assumption, intuitively it is very questionable that the trader could develop a trading system with a positive expected value without an edge and with a transaction cost. In fact, we can prove that without a transaction cost, no matter how the trader sets her Profit Limit $L$ and Stop Loss $S$, all her trades will have an expected value of 0.

**Conclusion 3.1:** If $\mu_0=0$ and $C=0$, then for any $i$, $E(O_i) = 0$ regardless of $S$, $L$, and $\Gamma$.

$$E(O) = E(X_{exit}) = \sum_{t=0}^{\Gamma} E(X_t) \cdot P(t = \text{exit})$$

$$E(X_t) = E(X_{t+1}) + E(\varepsilon_t) = E(X_{t+1}) = \ldots = E(X_0) = X_0 = 0 \quad \text{for} \quad \forall t \in (0, \Gamma)$$

Hence, $E(O) = 0$.

It is noteworthy to point out that $S$ and $L$ only affects $P(t = \text{exit})$ but not $E(X_0)$, which intuitively explains this conclusion.

With a positive $C$, then it is easy to see $E(O_i) = 0 - C = -C$. From here, it is also easy to see that $E(P) = E(\sum_{i=1}^{n} O_i) = \sum_{i=1}^{n} E(O_i) = \sum_{i=1}^{n} C = -nC$ (Conclusion 3.2).

This conclusion can also be verified by running the Matlab program `tradingRandom.m` with the parameter drift set to 0 (e.g. try running `tradingRandom(100, 0, 0.4, 0, 10, 100, 3, 100000)`).

As this conclusion suggests, in a flat market, the trader is always expected to lose no matter how she trades; and the more she trades, the more she is expected to lose. Thus, in such markets, traders are better off not trading at all.

**Part IV. Full model with the drift $\mu_0$ and the edge $\alpha$**

Now to complete the model, let us introduce the drift variable $\mu_0$ and the trader’s edge variable $\alpha$, and the model become much more complicated to solve. Again, we try to start with a simpler exception. If the trader forgets to set her Profit Limit and Stop Loss, or sets them to be too far from the original price $X_0$ so that either of them could be effectively reached within the trend length $\Gamma$, then what would happen? Mathematically, this problem could be expressed as: if $\max(X_0, X_1, \ldots X_f) < L$ and $\min(X_0, X_1, \ldots X_f) > -S$, what is $E(P)$?

In such a case, a trade will not be exited on a Profit Limit or on a Stop Loss. The only way left for a trade to be exited in this case is when $t$ arrives at the trend length $\Gamma$ and the stock is sold at $X_f$. Therefore, the whole strategy will consist of a single trade with an outcome $O_f = X_f - C$. Thus,

$$E(P) = E(O_f) = E(X_f - C) = E(X_f) - C,$$

since

$$E(X_f) = E(\sum_{t=1}^{\Gamma} \mu_t + \sum_{t=1}^{\Gamma} \varepsilon_t) = E(\sum_{t=1}^{\Gamma} \mu_t) + E(\sum_{t=1}^{\Gamma} \varepsilon_t) = \sum_{t=1}^{\Gamma} E(\mu_t) + \sum_{t=1}^{\Gamma} E(\varepsilon_t),$$
and $E(\mu) = \left(\frac{1}{2} + \alpha\right)\mu_0 + \left(\frac{1}{2} - \alpha\right)(-\mu_0), \ E(\varepsilon) = \frac{1}{2}\varepsilon_0 + \frac{1}{2}(-\varepsilon_0) = 0,$

we get $E(X_t) = \sum_{i=1}^{\Gamma} E(\mu_i) + \sum_{i=1}^{\Gamma} E(\varepsilon_i) = 2\alpha\mu_0 \cdot \Gamma + 0 \cdot \Gamma = 2\alpha\mu_0 \Gamma.$

Hence, we arrive that $E(P) = E(X_t) - C = 2\alpha\mu_0 \Gamma - C.$ \hspace{1cm} (Conclusion 4.1)

This formula illustrates the tradeoff of the strategy: how much a trader expected to gain from the size of the trend ($\mu_0 \Gamma$) and her probability advantage to benefit from that trend ($2\alpha$), versus her transaction cost ($-C$). We can verify this conclusion by running the Matlab program tradingRandom.m with the appropriate parameters (e.g. try running tradingRandom(100, 0.6, 0.4, 0.1, 10000, 10000, 3, 100000)).

Now finally, we could examine the effect of Profit Limit ($L$) and Stop Loss ($S$). Let us first assume that when a trader is stopped out, she immediately enters the position again. What would the trader’s expected profit be?

$$E(P) = E\left(\sum_{i=1}^{n} O_i\right) = \sum_{i=1}^{n} E(O_i) = \sum_{i=1}^{n} (L - C) + \sum_{i=1}^{n} (-S - C) + (X_t - C) \cdot P_r$$

$$= l(L - C) + s(-S - C) + (X_t - C) \cdot P_r = (lL - sS + P_rX_t) - (l + s + P_r)C$$

where $l$ denotes the number of trades that reaches $L$, $s$ denotes the ones that ends at $S$, and $P_r$ denotes the probability that there will be a residual trade (the last trade exited in between $L$ and $-S$ when the trend ends, it could be avoided when the payoff $X_t - C$ is clearly negative.

To calculate $l$, $s$, and $P_r$ is beyond the scope of this paper (interested readers are welcome to explore these values with tradingRandom.m). However, with one more reasonable assumption, we could approximate $E(P)$ above. That assumption is, $\mu_0$ and $\varepsilon_0$ are both small. This is a reasonable assumption in modeling stock market as stock prices change in very small time intervals (often a small fraction of second), and therefore each incremental change (the combined result of $\mu_t$ and $\varepsilon_t$) is also very small. In account of that, let us examine what happens when the trader exits her position at time $t$ due to either a Profit Limit or Stop Loss:

If trader exits her position at $t$ due to a Profit Limit $L$, then we know $X_{t-1} < L$ and $X_t = L$, so

$$X_t - L = X_{t-1} + \mu_t + \varepsilon_t - L = \mu_t + \varepsilon_t + (X_{t-1} - L) <= \mu_t + \varepsilon_t <= \mu_0 + \varepsilon_0.$$  

Similarly, we know when a trader exits at $t$ due to a Stop Loss $S$, then $X_{t-1} > -S$ and $X_t <= S$

$$(-S) - X_t = -S - (X_{t-1} + \mu_t + \varepsilon_t) = -\mu_t - \varepsilon_t + (-S - X_{t-1}) <= -\mu_t - \varepsilon_t <= \mu_0 + \varepsilon_0.$$  

Therefore, we see that if we replace $S$ or $L$ with $X_t$ (the price at exit time $t$), the error is bounded by $\pm(\mu_0 + \varepsilon_0)$, in other words, very small. \hspace{1cm} (Conclusion 4.3)

Also, we will use $P_r = l$ in this approximation, as we will have a very high probability to end up with a residual (small or big).

Hence, we can approximate the expected profit $E(P)$ by:

$$E(P) = E\left(\sum_{i=1}^{n} O_i\right) = \sum_{i=1}^{n} E(O_i) = \sum_{i=1}^{n} E(X_{\text{ith-exit}} - C) = \sum_{i=1}^{n} E(X_{\text{ith-exit}}) - nC$$

$$= \sum_{i=1}^{\Gamma} E(\mu_i) + \sum_{i=1}^{\Gamma} E(\varepsilon_i) - nC = E(\sum_{i=1}^{\Gamma} \mu_i) + E(\sum_{i=1}^{\Gamma} \varepsilon_i) - nC = E(X_t) - nC$$
\[ = 2 \alpha \mu_0 \Gamma - nC. \]  
\textbf{(Conclusion 4.4)}

Note that this is clearly worse off than the result we get from the no Profit Limit or Stop Loss strategy (Conclusion 4.1) as it only increase trading cost but does not improve trading gains.

Hence, we arrive our final conclusion: the theoretical optimal trading strategy in this model is to: **predict the direction of the trend, buy (or short-sell) the stock in the beginning according to the predicted trend direction, and exit the position when the trend ends.** This simple strategy gives the best expected profit of:

\[ E(P) = 2 \alpha \mu_0 \Gamma - C. \]  
\textbf{(Conclusion 4.5)}

**Part V. A review on assumptions and limitations of the model**

The final conclusion about optimal strategy (Conclusion 4.5) seems a little surprising. Those who are more familiar with stock trading will often hear the importance of Stop Losses and Profit Limits. To understand this discrepancy, it is important to realize that all conclusions are arrived under all the assumptions of this model discussed above. Some key assumptions I would like to re-emphasize are:

1) The magnitude of the drift \( \mu_0 \), the disturbance term \( \varepsilon_0 \), and the trader’s edge \( \alpha \) are all constants. Hence, the trader cannot “learn” from her gains or losses in this model, which eliminates an important function of Stop Losses and Profit Limits in real trading.

2) The trader trades continuously in this model. Immediately after the trader exits a position, she enters again. This assimilates more with Program/Algorithmic trading, while real human traders often take reflective breaks or looks for optimal enter points during their trading.

3) The drifts \( \mu_t \) and the disturbance \( \varepsilon_t \) are small. This is probably the most important assumption made (also probably the most doubtful one) to achieve Conclusion 4.5. Because of the small increment price change, Stop Loss’s function of protecting traders from sharp price movements against them is minimized. While in real world trading, a trader who sets a stop loss could be protected from a large difference between \( X_{exit} \) and \( S \).

These assumptions reveal some limitations of this model. However, this paper and the conclusions still serve reasonably well as an exploratory effort towards finding an optimal trading strategy.
Appendix: tradingRandom.m

function avg_profit = tradingRandom(trend_length, drift, disturbance, edge, stop, limit, cost, run_time)

profit = 0;

for r = 1:run_time
    x = 0;
    randD = rand(1);
    if (randD < (.5 + edge))
        direction = 1;
    else
        direction = -1;
    end;
    
    t = 0;
    trading = 1;
    while (t < trend_length)
        randE = randi(2)*2-3;
        x = x + direction * drift + randE * disturbance;
        if (x <= -stop)
            profit = profit - stop - cost;
            trading = 0;
        end;
        if (x >= limit)
            profit = profit + limit - cost;
            trading = 0;
        end;
        t = t + 1;
    end;

    if (trading == 1)
        profit = profit + x - cost;
    end;
end;

avg_profit = profit / run_time;
end