

Ergodic Markov Chains

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Definition

- A Markov chain is called an *ergodic chain* if it is possible to go from every state to every state (not necessarily in one move).
- Ergodic Markov chains are also called *irreducible*.
- A Markov chain is called a *regular* chain if some power of the transition matrix has only positive elements.

Example

- Let the transition matrix of a Markov chain be defined by

$$\mathbf{P} = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

- Then this is an ergodic chain which is not regular.

Example: Ehrenfest Model

- We have two urns that, between them, contain four balls.
- At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn.
- We choose, as states, the number of balls in the first urn.

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix} .$$

Regular Markov Chains

- Any transition matrix that has no zeros determines a regular Markov chain.
- However, it is possible for a regular Markov chain to have a transition matrix that has zeros.
- For example, recall the matrix of the Land of Oz

$$\mathbf{P} = \begin{array}{c} \text{R} \\ \text{N} \\ \text{S} \end{array} \begin{array}{ccc} \text{R} & \text{N} & \text{S} \\ \left(\begin{array}{ccc} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{array} \right) . \end{array}$$

Theorem. *Let \mathbf{P} be the transition matrix for a regular chain. Then, as $n \rightarrow \infty$, the powers \mathbf{P}^n approach a limiting matrix \mathbf{W} with all rows the same vector \mathbf{w} . The vector \mathbf{w} is a strictly positive probability vector (i.e., the components are all positive and they sum to one).*

Example

- For the Land of Oz example, the sixth power of the transition matrix \mathbf{P} is, to three decimal places,

$$\mathbf{P}^6 = \begin{array}{c} \text{R} \quad \text{N} \quad \text{S} \\ \text{R} \\ \text{N} \\ \text{S} \end{array} \begin{pmatrix} .4 & .2 & .4 \\ .4 & .2 & .4 \\ .4 & .2 & .4 \end{pmatrix} .$$

Theorem. *Let \mathbf{P} be a regular transition matrix, let*

$$\mathbf{W} = \lim_{n \rightarrow \infty} \mathbf{P}^n ,$$

let \mathbf{w} be the common row of \mathbf{W} , and let \mathbf{c} be the column vector all of whose components are 1. Then

- (a)** $\mathbf{wP} = \mathbf{w}$, and any row vector \mathbf{v} such that $\mathbf{vP} = \mathbf{v}$ is a constant multiple of \mathbf{w} .
- (b)** $\mathbf{Pc} = \mathbf{c}$, and any column vector \mathbf{x} such that $\mathbf{Px} = \mathbf{x}$ is a multiple of \mathbf{c} .

Definition: Fixed Vectors

- A row vector \mathbf{w} with the property $\mathbf{wP} = \mathbf{w}$ is called a *fixed row vector* for \mathbf{P} .
- Similarly, a column vector \mathbf{x} such that $\mathbf{Px} = \mathbf{x}$ is called a *fixed column vector* for \mathbf{P} .

Example

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$$w_1 + w_2 + w_3 = 1$$

and

$$(w_1 \quad w_2 \quad w_3) \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = (w_1 \quad w_2 \quad w_3) .$$

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$$\begin{aligned}w_1 + w_2 + w_3 &= 1 , \\(1/2)w_1 + (1/2)w_2 + (1/4)w_3 &= w_1 , \\(1/4)w_1 + (1/4)w_3 &= w_2 , \\(1/4)w_1 + (1/2)w_2 + (1/2)w_3 &= w_3 .\end{aligned}$$

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The solution is

$$\mathbf{w} = (.4 \quad .2 \quad .4) ,$$

Another method

- Assume that the value at a particular state, say state one, is 1, and then use all but one of the linear equations from $\mathbf{wP} = \mathbf{w}$.
- This set of equations will have a unique solution and we can obtain \mathbf{w} from this solution by dividing each of its entries by their sum to give the probability vector \mathbf{w} .

Example (cont'd)

- Set $w_1 = 1$, and then solve the first and second linear equations from $\mathbf{wP} = \mathbf{w}$.

$$\begin{aligned}(1/2) + (1/2)w_2 + (1/4)w_3 &= 1, \\ (1/4) + (1/4)w_3 &= w_2.\end{aligned}$$

- We obtain

$$(w_1 \quad w_2 \quad w_3) = (1 \quad 1/2 \quad 1) .$$

Equilibrium

- Suppose that our starting vector picks state s_i as a starting state with probability w_i , for all i .
- Then the probability of being in the various states after n steps is given by $\mathbf{wP}^n = \mathbf{w}$, and is the same on all steps.
- This method of starting provides us with a process that is called “stationary.”

Ergodic Markov Chains

Theorem. *For an ergodic Markov chain, there is a unique probability vector \mathbf{w} such that $\mathbf{wP} = \mathbf{w}$ and \mathbf{w} is strictly positive. Any row vector such that $\mathbf{vP} = \mathbf{v}$ is a multiple of \mathbf{w} . Any column vector \mathbf{x} such that $\mathbf{Px} = \mathbf{x}$ is a constant vector.*

The Ergodic Theorem

Theorem. *Let \mathbf{P} be the transition matrix for an ergodic chain. Let \mathbf{A}_n be the matrix defined by*

$$\mathbf{A}_n = \frac{\mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \dots + \mathbf{P}^n}{n + 1} .$$

Then $\mathbf{A}_n \rightarrow \mathbf{W}$, where \mathbf{W} is a matrix all of whose rows are equal to the unique fixed probability vector \mathbf{w} for \mathbf{P} .

Exercises

Which of the following matrices are transition matrices for regular Markov chains?

1. $\mathbf{P} = \begin{pmatrix} .5 & .5 \\ 1 & 0 \end{pmatrix}.$

2. $\mathbf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix}.$

3. $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

Consider the Markov chain with general 2×2 transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix} .$$

1. Under what conditions is \mathbf{P} absorbing?
2. Under what conditions is \mathbf{P} ergodic but not regular?
3. Under what conditions is \mathbf{P} regular?

Exercises ...

Find the fixed probability vector \mathbf{w} for the matrices in the previous exercise that are ergodic.