1. Finish up with homogeneous equations, learning when they have a nontrivial solution.
2. Describe the solution set homogeneous equation as the span of a finite set of vectors.
3. Describe the solution set of an inhomogeneous equation.
4. Use systems of linear equations to model economic behavior.
5. Use systems of linear equations to model street traffic.
A homogeneous system $Ax = 0$ always has at least one solution:

$x = 0$.

If we form the linear combination of the columns of $A$ with all weights equal to 0, we just get $0 + 0 + \ldots + 0 = 0$. So the question of whether or not $Ax = 0$ is consistent is kinda boring—it always is. The interesting question becomes: can we find $x \neq 0$ with $Ax = 0$?
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Definition

Let $Ax = 0$ be a homogeneous system of linear equations. The trivial solution to this system is $x = 0$. A solution $x$ to $Ax = 0$ with $x \neq 0$ is referred to as a non-trivial solution.
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**Definition**

Let $Ax = 0$ be a homogeneous system of linear equations.
A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution: $\mathbf{x} = \mathbf{0}$. If we form the linear combination of the columns of $A$ with all weights equal to 0, we just get $0 + 0 + \ldots + 0 = 0$. So the question of whether or not $A\mathbf{x} = \mathbf{0}$ is consistent is kinda boring—it always is.

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A homogeneous system $Ax = 0$ always has at least one solution: $x = 0$. If we form the linear combination of the columns of $A$ with all weights equal to 0, we just get $0 + 0 + \ldots + 0 = 0$. So the question of whether or not $Ax = 0$ is consistent is kinda boring—it always is.

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Example

Let \( A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \).
Trivial and nontrivial solutions: examples

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\[
A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
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When do non-trivial solutions exist?

Suppose that $Ax = 0$.
When do non-trivial solutions exist?

Suppose that $Ax = 0$. There is a nontrivial solution exactly when the solutions are not unique.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$.

Is there a nontrivial solution to $Ax = 0$?

Reduce the augmented matrix to echelon form:

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

Is every column a pivot column?

No, so the solution is not unique, and there is a nontrivial solution to $Ax = 0$.

In this example, $x = (-2, 1, 0)$ is a nontrivial solution.
Suppose that $A\mathbf{x} = \mathbf{0}$. There is a nontrivial solution exactly when the solutions are not unique. So we can check to see if there is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$ by reducing the system to echelon form, then looking at pivot columns.

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Describing planes with parameters

Let’s the solution set of the homogeneous system

\[ 2x - 4y - 8z = 0 \]

as the span of some set of vectors.
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Solve for basic in terms of free: \(x = 2y + 4z\).
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Solve for basic in terms of free: \( x = 2y + 4z \). A solution looks like

\[(x, y, z) = (2y + 4z, y, z) = y(2, 1, 0) + z(4, 0, 1).\]
Describing planes with parameters

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Solve for basic in terms of free: $x = 2y + 4z$. A solution looks like

$$(x, y, z) = (2y + 4z, y, z) = y(2, 1, 0) + z(4, 0, 1).$$

Thus the solution set is

$$\text{Span}\{(2, 1, 0), (4, 0, 1)\}.$$
In the previous example, every solution to the system

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Parametric vector equations

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**Example**

The system

\[x - y = 0\]

has the solution set \(\{(x, x)\} = \text{Span}\{(1, 1)\} \subset \mathbb{R}^2\).
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**Example**

The system

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has the solution set \( \{(x, x)\} = \text{Span}\{(1, 1)\} \subset \mathbb{R}^2 \). It has the parametric equation \( x = x(1, 1) \), where \( x \) is a scalar.
We have a pretty good idea of how to describe the solution set of $Ax = 0$ in parametric form.
We have a pretty good idea of how to describe the solution set of $Ax = 0$ in parametric form. What about describing the solution set of $Ax = b$ where $b \neq 0$?

Example

Describe the solution set of $Ax = b$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $b = (3, 6)$.

The augmented matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

Now write $x + 2y = 3$, solve for the basic variable $x$, to get $x = 3 - 2y, y$ free.

Solutions look like $(x, y) = (3 - 2y, y) = (3, 0) + y(-2, 1)$, where $y$ is any number.
Inhomogeneous solutions

We have a pretty good idea of how to describe the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric form. What about describing the solution set of $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$?

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Theorem

Let $A\mathbf{x} = \mathbf{b}$ be an inhomogeneous matrix equation, where $A$ is a $m \times n$ matrix and $\mathbf{b} \neq \mathbf{0}$. Suppose that $p$ is a particular solution to the system. Consider the homogeneous system $A\mathbf{z} = \mathbf{0}$. Then every other solution $\mathbf{w}$ of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{w} = p + \mathbf{v}$$

where $\mathbf{v}$ is some solution to $A\mathbf{z} = \mathbf{0}$. 
Theorem

Let \( Ax = b \) be an inhomogeneous matrix equation, where \( A \) is a \( m \times n \) matrix and \( b \neq 0 \). Suppose that \( p \) is a particular solution to the system. Consider the homogeneous system \( Az = 0 \). Then every other solution \( w \) of \( Ax = b \) has the form

\[
w = p + v
\]

where \( v \) is some solution to \( Az = 0 \). Conversely, if \( v \) is some solution to \( Az = 0 \), then \( p + v \) is a solution to \( Ax = b \).
Theorem

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Proof.

Suppose that $Aw = b = Ap$. 

Dan Crytser  Lecture 5: Homogeneous, inhomogeneous, solution sets. Applications
Solutions of inhomogeneous equations

Theorem

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Proof.

Suppose that $Aw = b = Ap$. Then we can subtract to obtain

$$A(w - p) = b - b = 0.$$

So $w - p = v$ for some $v$ a solution of $Az = 0$
Proof.

Now suppose that \( p \) is a particular solution to \( Ax = b \) and \( Av = 0 \).
Proof.

Now suppose that $\mathbf{p}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{0}$. Then

$$A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$
Proof.

Now suppose that \( p \) is a particular solution to \( Ax = b \) and \( Av = 0 \). Then

\[
A(p + v) = Ap + Av = b + 0 = b.
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Thus \( p + v \) is a solution to \( Ax = b \) as well.
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The previous theorem shows that to describe the solutions of \( Ax = b \) takes two steps:
Proof.

Now suppose that \( \mathbf{p} \) is a particular solution to \( A\mathbf{x} = \mathbf{b} \) and \( A\mathbf{v} = \mathbf{0} \). Then

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The previous theorem shows that to describe the solutions of \( A\mathbf{x} = \mathbf{b} \) takes two steps:

1. find one particular solution \( \mathbf{p} \), that is a vector \( \mathbf{p} \) such that \( A\mathbf{p} = \mathbf{b} \);
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Now suppose that \( p \) is a particular solution to \( Ax = b \) and \( Av = 0 \). Then

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The previous theorem shows that to describe the solutions of \( Ax = b \) takes two steps:

1. find one particular solution \( p \), that is a vector \( p \) such that \( Ap = b \);
2. describe all solutions to the affiliated homogeneous system \( Az = 0 \).
Proof.

Now suppose that $\mathbf{p}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{0}$. Then

$$A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$ 

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The previous theorem shows that to describe the solutions of $A\mathbf{x} = \mathbf{b}$ takes two steps:

1. find one particular solution $\mathbf{p}$, that is a vector $\mathbf{p}$ such that $A\mathbf{p} = \mathbf{b}$;

2. describe all solutions to the affiliated homogeneous system $A\mathbf{z} = \mathbf{0}$

The solutions set is then $\{\mathbf{p} + \mathbf{v} : A\mathbf{v} = \mathbf{0}\}$.
Example: parametric form for inhomogeneous solutions

Write all solutions to

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 6 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
3 \\
\end{bmatrix}.
\]

We see that the first column equals the vector \([1 \ 3]\), so a particular solution is given by \(x = 1, y = 0, z = 0\).

Now we describe the solutions to the associated homogeneous equation

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 6 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}.
\]

The augmented matrix is

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
3 & 6 & 4 & 0 \\
\end{bmatrix}
\sim 
\begin{bmatrix}
1 & 2 & 1 & 0 \\
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We see that the first column equals the vector \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \), so a particular solution is given by \( x = 1, y = 0, z = 0 \). Now we describe the solutions to the associated homogeneous equation

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 6 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}.
\]

The augmented matrix is

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
3 & 6 & 4 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]
Example: parametric form for inhomogeneous solutions

Write all solutions to

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 6 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
3 \\
\end{bmatrix}.
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We see that the first column equals the vector \[\begin{bmatrix} 1 \\ 3 \end{bmatrix}\], so a particular solution is given by \(x = 1, y = 0, z = 0\). Now we describe the solutions to the associated homogeneous equation

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\]
Solving the previous homogeneous equation yields $x = -2y$, $z = 0$, $y$ free.
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$$\text{Span}\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}.$$
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$$\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$ 

A particular solution to the inhomogeneous equation is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

Thus the general parametric form of the solution to the inhomogeneous is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

where $t$ is allowed to be any real number.
Example, ctd.

Solving the previous homogeneous equation yields \( x = -2y, \ z = 0, \) \( y \) free. The solution set of the associated homogeneous equation is 

\[
\text{Span}\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}.
\]

A particular solution to the inhomogeneous equation is 

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Thus the general parametric form of the solution to the inhomogeneous is 

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}
\]

where \( t \) is allowed to be any real number.
Now we’re going to look at some applications of linear systems: economics and street traffic.
Most applications of linear algebra come from modeling some quantity which changes hands of locations with a set of linear equations.
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The first example of this we shall see comes from economics. Many economists divide the economy of a city, province, or nation into sectors. Examples of these could include: coal, electricity, steel etc. You measure the output of a sector in dollars. The sectors use their own output and the output of the other sectors: steel needs coal, producing coal requires electricity, electric plants need steel, etc.
Suppose that you are studying an economy in which there are just these three sectors (C,E,S).
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<td>E</td>
<td>S</td>
<td>C</td>
</tr>
<tr>
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<td>.4</td>
<td>.6</td>
<td></td>
</tr>
<tr>
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We denote the output of the three sectors by $p_C, p_E, p_S$. 
In a balanced economy the amount each sector spends equals the amount it produces.

For example, if $C$ produces $\$p_C$ and spends $\$0.4E + \$0.6S$, so $p_C = 0.4p_E + 0.6p_S$ if the economy is to be balanced.

Similarly, $p_E = 0.6p_C + 0.1p_E + 0.2p_S$ and $p_S = 0.4p_C + 0.5p_E + 0.2p_S$.

Rewriting these in linear form we obtain

$$p_C - 0.4p_E - 0.6p_S = 0$$

$$-0.6p_C + 0.9p_E - 0.2p_S = 0$$

$$-0.4p_C - 0.5p_E + 0.8p_S = 0$$

as our system of linear equations.

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For example, if C produces \( p_C \) dollars and spends \( .4 p_E + .6 p_S \), then if the economy is to be balanced, \( p_C = .4 p_E + .6 p_S \). Similarly, \( p_E = .6 p_C + .1 p_E + .2 p_S \) and \( p_S = .4 p_C + .5 p_E + .2 p_S \). Rewriting these in linear form we obtain:

\[
\begin{align*}
p_C - .4 p_E - .6 p_S &= 0 \\
-.6 p_C + .9 p_E - .2 p_S &= 0 \\
-.4 p_C - .5 p_E + .8 p_S &= 0
\end{align*}
\]
as our system of linear equations.
In a balanced economy the amount each sector spends equals the amount it produces. For example C produces $p_C$ dollars and spends $0.4p_E + 0.6p_S$. So

$$p_C = 0.4p_E + 0.6p_S$$

if the economy is to be balanced.
In a balanced economy the amount each sector spends equals the amount it produces. For example C produces $p_C$ dollars and spends $0.4p_E + 0.6p_S$. So

$$p_C = 0.4p_E + 0.6p_S$$

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\[ p_C - .4p_E - .6p_S = 0 \]
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We solve this by row reduction on the augmented matrix

\[
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
-.6 & .9 & -.2 & 0 \\
-.4 & -.5 & .8 & 0
\end{bmatrix}
\]

The solution is therefore

\[ p_C = .94p_S, \]
\[ p_E = .85p_S, \]
\[ \text{and } p_S \text{ free.} \]

The price vector is

\[ p = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix} \]

with \( p_S \) free.

Also need \( p_S \geq 0 \).
\[
p_C - .4p_E - .6p_S = 0 \\
-.6p_C + .9p_E - .2p_S = 0 \\
-.4p_C - .5p_E + .8p_S = 0
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We solve this by row reduction on the augmented matrix

\[
\begin{bmatrix}
1 & - .4 & - .6 & 0 \\
-.6 & .9 & - .2 & 0 \\
-.4 & - .5 & .8 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & - .94 & 0 \\
0 & 1 & - .85 & 0 \\
0 & 0 & 0 & 0
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\[ \begin{align*}
  p_C - .4p_E - .6p_S &= 0 \\
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The solution is therefore \( p_C = .94p_S, \ p_E = .85p_S, \) and \( p_S \) free.
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\[
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  p_S
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  .85 \\
  1
\end{bmatrix}
\]

with \( p_S \) free.
Economics, ctd.

\[ p_C - .4p_E - .6p_S = 0 \]
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\[
\begin{bmatrix}
1 & -4 & -6 & 0 \\
-6 & 9 & -2 & 0 \\
-4 & -5 & 8 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -94 & 0 \\
0 & 1 & -85 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The solution is therefore \( p_C = .94p_S \), \( p_E = .85p_S \), and \( p_S \) free.

The price vector is

\[
p = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}
\]

with \( p_S \) free. Also need \( p_S \geq 0 \).
The mathematical framework of *networks* is useful in many different contexts.
The mathematical framework of *networks* is useful in many different contexts. The following definition you don’t need to know by heart.

**Definition**

A network is a collection of nodes joined by branches which connect one node to another, along with numbers called flow amounts (or weights) through each branch.

**Example**

Streets can be used as an example of networks:

1. the streets are the branches, with direction
2. the nodes are the intersections
3. the hourly traffic following along a street in a given direction is the flow weight.
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2. the nodes are the intersections
3. the hourly traffic following along a street in a given direction is the flow weight.
Network flow: example

The nodes are A, B, C, D.
The flow into each node has to equal the flow out of each node.

Node (intersection) | Flow in | Flow out
---|---|---
A | 300 + 500 = \(x_1 + x_2\) | 100 + \(x_2\) = 300 + \(x_3\)
B | \(x_2\) + \(x_4\) = 300 + \(x_3\) | \(x_4\) = \(x_5\)
C | 100 + 400 = \(x_4 + x_5\) | \(x_5\) = 600
D | \(x_1\) + \(x_5\) = 600 | 400 + \(x_5\) = \(x_4\)
The nodes are A,B,C,D.
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We have the balanced flow equations

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We also need that the flow into the system $(500 + 300 + 100 + 400)$ equals the flow out $(300 + x_3 + 600)$. 
Traffic flow, ctd.

We have the balanced flow equations

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We also need that the flow into the system 
(500 + 300 + 100 + 400) equals the flow out (300 + $x_3$ + 600). We simplify and combine all of this into a system of equations:

\[
\begin{align*}
    x_1 + x_2 & = 800 \\
    x_2 - x_3 + x_4 & = 300 \\
    x_4 + x_5 & = 500 \, . \\
    x_1 + x_5 & = 600 \\
    x_3 & = 400 
\end{align*}
\]
We have the balanced flow equations

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\[
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\]
Traffic flow, ctd.

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  x_4 + x_5 &= 500 \\
  x_1 + x_5 &= 600 \\
  x_3 &= 400
\end{align*} \]

If we solve this system with row reduction we get the solution set

\[ \begin{align*}
  x_1 &= 600 - x_5 \\
  x_2 &= 200 + x_5 \\
  x_3 &= 400 \\
  x_4 &= 500 - x_5 \\
  x_5 & \text{is free}
\end{align*} \]

Again, real world constraints make the solution set smaller. \( x_4 \) cannot be negative because there cannot be a negative number of cars passing through a branch. So \( 0 \leq x_5 \leq 500 \).
Traffic flow, ctd.

\[
\begin{align*}
  x_1 + x_2 &= 800 \\
  x_2 - x_3 + x_4 &= 300 \\
  x_4 + x_5 &= 500 \\
  x_1 + x_5 &= 600 \\
  x_3 &= 400
\end{align*}
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\[
\begin{cases}
  x_1 = 600 - x_5 \\
  x_2 = 200 + x_5 \\
  x_3 = 400 \\
  x_4 = 500 - x_5 \\
  x_5 \text{ is free}
\end{cases}
\]

Again, real world constraints make the solution set smaller. \( x_4 \) cannot be negative because there cannot be a negative number of cars passing through a branch. So \( 0 \leq x_5 \leq 500 \).
Traffic flow, ctd.

\begin{align*}
    x_1 + x_2 &= 800 \\
    x_2 - x_3 + x_4 &= 300 \\
    x_4 + x_5 &= 500 \\
    x_1 + x_5 &= 600 \\
    x_3 &= 400
\end{align*}

If we solve this system with row reduction we get the solution set

\[
\begin{cases}
    x_1 &= 600 - x_5 \\
    x_2 &= 200 + x_5 \\
    x_3 &= 400 \\
    x_4 &= 500 - x_5 \\
    x_5 &\text{is free}
\end{cases}
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What this is used for

(You don’t need to know this for HW, exams, etc. ): The previous set-up will be familiar to anyone who has studied *operations research* (OR).

In OR we want to maximize or minimize some linear objective function of the variables, like

\[ f(x_1, x_2, x_3, x_4) = 2x_1 + x_2 - x_3 + 7x_4. \]

The idea is that the first step describes all the traffic configurations as a higher-dimensional object called the set of feasible solutions.

There are methods you can use the find the minimum or maximum value of \( f \), most famously the simplex algorithm invented by George Dantzig, which uses the geometry of the set of feasible solutions to efficiently find the optimal solution.
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