Lecture 8: The matrix of a linear transformation. Applications

Danny W. Crytser

April 7, 2014
Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

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\[ A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \\ 3x_2 \end{bmatrix} = T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}). \]

Thus \( T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = Ax \) for all \( x \in \mathbb{R}^2 \).

There's a fancy term for the matrix we've cooked up. Definition: If \( T: \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation and \( e_1, e_2, \ldots, e_n \) are the standard basis vectors in \( \mathbb{R}^n \), then the matrix \( A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \) which satisfies \( T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = Ax \) for all \( x \in \mathbb{R}^n \) is called the standard matrix for \( T \).
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Matrices and visualization of linear transformations

There are a few different types of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ that we can describe with words ("rotate the plane counterclockwise by $\pi/2$") and then we get the matrix just by tracking the image of the basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
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**TABLE 1** Reflections

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Image of the Unit Square</th>
<th>Standard Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection through the $x_1$-axis</td>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
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### TABLE 2  Contractions and Expansions

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Image of the Unit Square</th>
<th>Standard Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal contraction and expansion</td>
<td></td>
<td>$\begin{bmatrix} k &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
</tr>
</tbody>
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Shear transformations

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<tr>
<th>Transformation</th>
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<tbody>
<tr>
<td>Horizontal shear</td>
<td><img src="image1.png" alt="Image" /></td>
<td>$\begin{bmatrix} 1 &amp; k \ 0 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>Vertical shear</td>
<td><img src="image2.png" alt="Image" /></td>
<td>$\begin{bmatrix} 1 &amp; 0 \ k &amp; 1 \end{bmatrix}$</td>
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A function \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be **onto** if the range is the codomain, that is, for each vector \( y \in \mathbb{R}^m \) there is at least one \( x \in \mathbb{R}^n \) with \( T(x) = y \).
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Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x_1, x_2) = (x_1, 0)$. Then $T$ is not onto: the range is the $x$-axis, an object in mathematics noteworthy for not being the entire plane.
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The quality of being onto has to do with existence of solutions: a linear transformation $T$ given by $T(x) = Ax$ is onto if $Ax = b$ is consistent for all $b \in \mathbb{R}^m$. 

Reviewing the following theorem allows us to describe onto linear transformations with echelon forms.

We already saw a version of this theorem

**Theorem**

Let $A$ be an $m \times n$ matrix. Then $Ax = b$ is consistent for all $b \in \mathbb{R}^m$ if and only if every row in the echelon form of $A$ (not augmented) has a nonzero entry.

We can reformulate it in terms of linear transformations.

**Theorem**

Let $T(x) = Ax$. Then $T$ is onto if and only if every row in the echelon form of $A$ (non-augmented) has a nonzero entry.

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A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if $T(x) = T(x')$ implies $x = x'$ for vectors $x, x' \in \mathbb{R}^n$. That is, $T$ is one-to-one if two vectors in the domain have the same image under $T$ only when they are equal.
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We can summarize all of this in one biggish theorem:

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Let $A$ be an $m \times n$ matrix. The linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $T(x) = Ax$ is
1) onto if and only if every row of the echelon form of $A$ has a pivot
2) one-to-one if and only if every column of the echelon form of $A$ has a pivot

You can see that the only way that $T$ can be both onto and one-to-one is if $m = n$. 

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Lecture 8: The matrix of a linear transformation. Applications
Onto/one-to-one: echelon form

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Lecture 8: The matrix of a linear transformation. Applications
We can summarize all of this in one biggish theorem:

**Theorem**

Let $A$ be an $m \times n$ matrix. The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ is

1. onto if and only if every row of the echelon form of $A$ has a pivot and only if the columns of $A$ span $\mathbb{R}^m$
2. one-to-one if and only if every column of the echelon form of $A$ has a pivot and only if the columns of $A$ are linearly independent

You can see that the only way that $T$ can be both onto and one-to-one is if $m = n$. 
Example

Let \( A = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 4 & 2 \end{bmatrix} \) and define \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) by \( T(x) = Ax \).
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APPLICATIONS
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3. A voltage source is positive for a loop if the current flows from the positive (long) terminal to the negative (short) terminal.
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![Diagram](image)

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![Diagram of a circuit](image)

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Passing from \( A \) to \( B \) the sum of the voltage drops is \( 3I_1 - 3I_2 \). (Notice that the voltage source in the first loop is positive.)
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Remember: when adding voltage sources you have to check to see if they’re positive (current runs positive terminal to negative terminal) or negative (vice versa).
Kirchhoff: example

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Let’s look at this. First loop: Voltage source = 30. Voltage drop in first loop is $4I_1 + (3I_1 - 3I_2) + 4I_1 = 11I_1 - 3I_2$. Must equal voltage sources in first loop = 30. So the equation for the first loop is $11I_1 - 3I_2 = 30$. 

**FIGURE 1**
Kirchhoff: example

\[ 11I_1 - 3I_2 = 30 \]  \hspace{1cm} (1)
\[ -3I_1 + 6I_2 - I_3 = 5 \]  \hspace{1cm} (2)
\[ -I_2 + 3I_3 = -25 \]  \hspace{1cm} (3)

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Dan Crytser  
Lecture 8: The matrix of a linear transformation. Applications
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Has a unique solution: \( I_1 = 3 \) amps, \( I_2 = 1 \) amps, \( I_3 = -8 \) amps.
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Has a unique solution: \(l_1 = 3\) amps, \(l_2 = 1\) amps, \(l_3 = -8\) amps. The negative \(l_3\) answer says that the current flows clockwise in loop 3.
In many situations you will be measuring some system and all the information about the system at time $k$ will be contained in some vector $\mathbf{x}_k$. 

**Definition**

Suppose your data take the form of vectors $\mathbf{x}_k \in \mathbb{R}^n$, where $k = 0, 1, 2, ...$ If there is an $n \times n$ matrix $\mathbf{A}$ such that $\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0$, $\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1$ and generally $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$ (*), we say that equation (*) is a linear difference equation (some people call it a recursion relation, because it gives new measurement in terms of the old measurement).
In many situations you will be measuring some system and all the information about the system at time $k$ will be contained in some vector $\mathbf{x}_k$. (Could be age, salary, population, microbe count, whatever).

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You can study population dynamics using difference equations.
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where $r_0$ is the population in the city in year 0 and $s_0$ is the population in the suburb in year 0.
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Difference equations and population

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where $r_0$ is the population in the city in year 0 and $s_0$ is the population in the suburb in year 0. The vectors $x_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}$, $x_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}$ record the population distribution in year 1, year 2, etc.
Let's say that in any one year 95 percent of city people remain in the city and 5 percent of city people go to the suburb.

\[ r_{1} = 0.95r_{0} + 0.03s_{0}, \]
\[ s_{1} = 0.5r_{0} + 0.97s_{0}. \]
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and, in general,

\[ x_k = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} x_{k-1}. \]
The Transition Matrix

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If \( A \) is the matrix above, then we can write \( \mathbf{x}_k = A\mathbf{x}_{k-1} \).
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If A is the matrix above, then we can write \( x_k = A x_{k-1} \). You can use this to predict the future.