Traces on graph algebras

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Definition

A $C^*$-algebra is a complex $*$-algebra $A$ with norm $\| \cdot \|$ such that

1. $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in A$
2. $A$ is complete with respect to the norm $\| \cdot \|$.
3. $\|a^* a\| = \|a\|^2$ for any $a \in A$.  

Example: The complex numbers $\mathbb{C}$ form a $C^*$-algebra, with $z^* = z$.

Example: The matrices $M_n(\mathbb{C})$ (with $*$ given by conjugate transpose).

Example: Bounded operators $B(H)$ ($*$ is adjoint).

Example: If $X$ is a locally compact Hausdorff space, then $C_0(X) := \{f : X \to \mathbb{C} | f$ is continuous and vanishes at $\infty\}$ is a $C^*$-algebra under the $\|\cdot\|_\infty$-norm and pointwise operations.
Definition

A *C*-algebra is a complex *-algebra $A$ with norm $\| \cdot \|$ such that

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**Definition**

A *$C^*$-algebra* is a complex $\ast$-algebra $A$ with norm $|| \cdot ||$ such that:

1. $||ab|| \leq ||a|| ||b||$ for any $a, b \in A$
2. $A$ is complete with respect to the norm $|| \cdot ||$
3. $||a^*a|| = ||a||^2$ for any $a \in A$.

*Example*: The complex numbers $\mathbb{C}$ form a $C^*$-algebra, with $z^* = \overline{z}$. The matrices $M_n(\mathbb{C})$ (with $\ast$ given by conjugate transpose).
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Example: The complex numbers \(\mathbb{C}\) form a \(C^*\)-algebra, with \(z^* = \overline{z}\). The matrices \(M_n(\mathbb{C})\) (with \(*\) given by conjugate transpose). Bounded operators \(B(H)\) (\(*\) is adjoint).
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C*-algebras generated by partial isometries

Definition

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$C^*$-algebras generated by partial isometries have a long history.

\textbf{Theorem (Coburn, '67)}

\textit{If $A$ is generated by an element $t$ satisfying $t^*t = 1$ and $tt^* \preceq 1$, then $A \cong \mathcal{T}$, the Toeplitz algebra.}
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**Theorem (Cuntz, ’77)**

*If $A$ is generated by elements $s, t$ satisfying*

$$s^*s = t^*t = ss^* + tt^* = 1$$

*then $A \cong \mathcal{O}_2$, the Cuntz algebra.*
Directed graphs

**Definition**

A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$, where $E^0$ and $E^1$ are (countable) sets and $r, s : E^1 \rightarrow E^0$ are functions called the *range* and *source* map, respectively.
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(All the graphs in this talk will be directed, so we might start just referring to them as graphs.) You can visualize a directed graph by drawing a point in the plane for each $v \in E^0$ and drawing for each edge $e \in E^1$ an arrow from $s(e)$ to $r(e)$. 

\[ \begin{array}{c}
\vdots \\
\bullet & \blackleft & \blackleft & \bullet \\
\downarrow & \blackleft & \blackleft & \downarrow \\
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\downarrow & \blackleft & \blackleft & \downarrow \\
\end{array} \]
Operator algebraists like graphs because they give us a standard way to study a wide class of $C^*$-algebras generated by partial isometries. The basic idea is that you keep track of the relations between the generators using the edge matrix of a directed graph.
Graph \( C^* \)-algebras

**Definition**

Given a directed graph \( E = (E^0, E^1, r, s) \) the *graph algebra* \( C^*(E) \) is the universal \( C^* \)-algebra generated by a family \( \{ s_e, p_v : e \in E^1, v \in E^0 \} \), where the \( p_v \) are mutually orthogonal projections and the \( s_e \) are partial isometries with mutually orthogonal range projections satisfying

1. \( s_e^* s_e = p_{s(e)} \)

2. \( s_e s_e^* \leq p_{r(e)} \)

3. \( p_v = \sum_{r(e)=v} s_e s_e^* \) if \( 0 < |r^{-1}(v)| < \infty \).
Graph algebras

If $E$ is the graph

\[ \begin{array}{c}
  \text{V} \\
  \bullet \\
  \text{e} \\
  \bullet \\
  \text{W}
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then you can show that $\mathbb{C}^*\left(\mathbb{M}_2(\mathbb{C})\right)$. 

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Traces on graph algebras
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Then $C^*(E) \cong \mathcal{T}$, the Toeplitz algebra.
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Graph algebras

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We aim to characterize two $C^*$-algebraic properties for graph algebras. First, we determine which graphs yield *continuous-trace* graph algebras. Then we examine existing theorems determining which graphs yield *stable* graph algebras.
Part I: Continuous-trace graph algebras
Hausdorff spectrum

The set of unitary equivalence classes of irreducible representations of a $C^*$-algebra $A$ forms a topological space called the spectrum of $A$, denoted by $\hat{A}$. This can be a poorly-behaved topological space.

**Example**

The spectrum of $B(H)$ is uncountable and non-Hausdorff.

Many people have studied various topological aspects of the spectrum. Goehle determined when a suitably nice graph $E$ yields a graph algebra with Hausdorff spectrum.
If $A$ has Hausdorff spectrum then for any point $t = [\pi]$ in the spectrum and any element $a \in A$, you can consider 
$a(t) = a + \ker \pi \in A/ \ker \pi$. Since $\hat{A}$ is Hausdorff, this has a well-defined rank.
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**Definition**

Let $A$ be a $C^*$-algebra with Hausdorff spectrum. Then $A$ has *continuous trace* if for every point $t \in \hat{A}$, there is a neighborhood $U$ of $t$ and an element $a \in A$ such that $a(s)$ is a rank-one projection for all $s \in U$. The upshot of this is that continuous-trace $C^*$-algebras act like “locally trivial non-commutative fiber bundles.” These algebras are well-studied and have nice representation theory.
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Let $A$ be a C*-algebra with Hausdorff spectrum. Then $A$ has **continuous trace** if for every point $t \in \hat{A}$, there is a neighborhood $U$ of $t$ and an element $a \in A$ such that $a(s)$ is a rank-one projection for all $s \in U$.

The upshot of this is that continuous-trace C*-algebras act like “locally trivial non-commutative fiber bundles.” These algebras are well-studied and have nice representation theory.
Example: Let $X$ be a locally compact Hausdorff space and let $A = C_0(X, \mathcal{K})$ denote the set of all continuous functions from $X$ to $\mathcal{K}$ which vanish at infinity. Then $A$ has continuous trace.
Continuous-trace $C^*$-algebras

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Example: Let

$$A = \{ f : [0, 1] \to M_2(\mathbb{C}) : f \text{ is continuous, } f(0) = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \}. $$

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Example: Let $X$ be a locally compact Hausdorff space and let $A = C_0(X, K)$ denote the set of all continuous functions from $X$ to $K$ which vanish at infinity. Then $A$ has continuous trace.

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Then $A$ has continuous trace. We characterize those graphs which yield continuous-trace graph algebras.
In order to determine when a graph $E$ yields a continuous-trace graph algebra, we use groupoids. A groupoid $G$ is a set along with

1. a subset $G^{(2)} \subset G \times G$ of composable pairs;
2. an associative operation $G^{(2)} \rightarrow G$ written $(\alpha, \beta) \rightarrow \alpha \beta$ called composition;
3. a map $G \rightarrow G$ written $\gamma \rightarrow \gamma^{-1}$ called inversion which allows cancellation on the left and right.
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There is no longer any identity element in a groupoid but there are “partial identities” called units. A *unit* of $G$ is an element $u$ such that $u = u^2 = u^{-1}$. In general there are many units; they form the *unit space* of $G$, denoted by $G^{(0)}$. 
Let \( r : G \to G^{(0)} \) be given by \( r(\gamma) = \gamma \gamma^{-1} \) and \( s : G \to G^{(0)} \) be given by \( s(g) = \gamma^{-1} \gamma \). Then \( r \) and \( s \) are referred to as the range and source maps of \( G \).
Groupoids

Let $r : G \to G^{(0)}$ be given by $r(\gamma) = \gamma \gamma^{-1}$ and $s : G \to G^{(0)}$ be given by $s(g) = \gamma^{-1} \gamma$. Then $r$ and $s$ are referred to as the range and source maps of $G$. These maps give a nice description of the composable pairs: $(\gamma, \gamma') \in G^{(2)}$ if and only if $s(\gamma) = r(\gamma')$. We can visualize an element in $G$ as an arrow from its source to its range.
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A topological groupoid is a groupoid with a topology that makes the operations continuous.
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**Definition**

A topological groupoid is étale if the range and source maps are local homeomorphisms.

If \( G \) is étale then \( r^{-1}(u) \) and \( s^{-1}(u) \) are discrete for any \( u \in G^{(0)} \).
Groupoids are interesting for many reasons, but we mostly use them to construct $C^*$-algebras. If $G$ is a second countable locally compact Hausdorff étale groupoid, then we can define operations on $C_c(G)$ by

$$f \ast g(\gamma) = \sum_{\alpha \beta = \gamma} f(\alpha)g(\beta)$$

and

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These operations make $C_c(G)$ into a $\ast$-algebra. You can give $C_c(G)$ a norm by taking a supremum over certain representations into $C^*$-algebras. Completing yields the groupoid $C^*$-algebra $C^*(G)$. 
If $E$ is a directed graph then there is an affiliated *path groupoid* $G_E$. The elements of $G_E$ are built out of *infinite paths*: sequences of edges $e_1e_2\ldots$ with $s(e_i) = r(e_{i+1})$. The collection of such paths is denoted $E^\infty$. There is for any integer $k \geq 0$ a *shift map* on $E^\infty$: 

$$\sigma^k(e_1e_2\ldots) = e_{k+1}e_{k+2}\ldots.$$ 

**Definition** 

The path groupoid $G_E \subset E^\infty \times \mathbb{Z} \times E^\infty$ consists of all triples $(x, n, y)$ such that there exist $p, q$ with $\sigma^p x = \sigma^q y$ and $p - q = n$.

The unit space of $G_E$ is identified with $E^\infty$. 
Let $E$ be the graph

If $x = \mu_1\mu_2\mu_3\mu_4\mu_5\gamma_1\gamma_2\gamma_3 \ldots$ and $y = \xi_1\xi_2\gamma_3 \ldots$, then the triple $(x, 5, y)$ belongs to $G_E$ because $\sigma^7 x = \sigma^2 y$. 

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Traces on graph algebras
The path groupoid carries a natural topology with basis consisting of all sets of the form

\[ Z(\alpha, \beta) = \{(az, |\alpha| - |\beta|, \beta z) : \alpha, \beta \in E^*, r(z) = s(\alpha) = s(\beta)\}, \]

where \( E^* \) denotes the finite path space. This topology makes \( G_E \) into a locally compact Hausdorff second countable étale groupoid, so we can construct its groupoid \( C^* \)-algebra.

**Theorem (KPRR, ’98)**

*If \( E \) is a row-finite graph with no sources, then there is an isomorphism \( C^*(E) \rightarrow C^*(G_E) \) which carries the edge partial isometry \( s_e \) onto the characteristic function \( \chi_{Z(e,s(e))} \in C_c(G_E) \subset C^*(G_E) \).*
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Now we can study \( C^*(E) \) by studying \( G_E \): we look for conditions on a groupoid that yield a continuous-trace algebra, and then determine how \( E \) has to behave for \( G_E \) to satisfy those conditions.
Definition

Let $G$ be a groupoid. If $u \in G^{(0)}$, the *stabilizer subgroup of $u$* is the set $G(u) = \{g \in G : r(g) = u = s(g)\}$. A groupoid is *principal* if $G(u) = \{u\}$ for each $u \in G^{(0)}$.

If $G$ is a groupoid then there is a principal groupoid $R = \{(u, v) \in G^{(0)} \times G^{(0)} : (u, v) = (r(g), s(g)) \text{ for some } g \in G\}$ and a groupoid homomorphism $\pi : G \to R$ given by $\pi(g) = (r(g), s(g))$. We call this the *orbit groupoid* of $G$. If $G$ is a nice topological groupoid then $R$ is a topological groupoid carrying the quotient topology.
Any groupoid acts on its unit space via the formula

\[ g \cdot s(g) = r(g). \]

We say that a topological groupoid acts *properly* on its unit space if the map

\[ \Phi : G \to G^{(0)} \times G^{(0)} \]

given by \( g \to (r(g), s(g)) \) is proper.
A topological groupoid $G$ has *continuously varying stabilizers* if the map $u \to G(u)$ which assigns to each unit its stabilizer subgroup is continuous. (Here the set of stabilizer subgroups is topologized with the *Fell topology.*)
Now we can say when a groupoid yields a $C^*$-algebra with continuous trace.

**Theorem (MRW, ’96)**

Suppose that $G$ is a second countable locally compact Hausdorff groupoid with unit space $G^{(0)}$, abelian stabilizers, and Haar system. Then $C^*(G)$ has continuous trace if and only if

1. the stabilizer map $u \mapsto G(u)$ is continuous, and
2. the orbit groupoid $R$ acts properly on its unit space $R^{(0)} = G^{(0)}$. 

As $C^*(G_E) \cong C^*(E)$ (when $E$ is nice), determining which graphs yield continuous-trace graph algebras is reduced to the question of determining which graphs yield path groupoids satisfying the above conditions.
Continuous-trace groupoid algebras

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Continuous-trace graph algebras

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An *entrance to a cycle* $\lambda = e_1 \ldots e_n$ is an edge $f$ with $r(f) = r(e_k)$ for some $k$ such that $f \neq e_k$.
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Here’s a simple example of an entrance to a cycle.
 Proposition (Goehle, ’13)

Let $E$ be a row-finite graph with no sources. Then $G_E$ has continuously varying stabilizers if and only if no cycle of $E$ has an entrance.

Thus the only thing that remains is to find conditions on $E$ that ensure the orbit groupoid $R_E$ acts properly on $E^\infty$. 
Continuous-trace graph algebras

Let $v, w$ be vertices. An *ancestry pair* is a pair of edges $(\lambda, \mu) \in E^* \times E^*$ such that

1. $r(\lambda) = v, r(\mu) = w$
2. $s(\mu) = s(\lambda)$,
3. neither path contains a cycle.

An ancestry pair is *minimal* if there is no factorization $(\lambda, \mu) = (\lambda', \nu, \mu')$ for another ancestry pair $(\lambda', \mu')$.

**Definition**

A graph has *finite ancestry* if given any two vertices $v$ and $w$ there are only finitely many minimal ancestry pairs for $v$ and $w$. 
Here \((\gamma_1 \gamma_2 \gamma_3, \xi \gamma_3)\) is an ancestry pair which is not minimal. The ancestry pair \((\gamma_2, \xi_2)\) is minimal.
Theorem (C., ’13)

Let $E$ be a row-finite graph with no sources. Then $C^*(E)$ has continuous trace if and only if

1. no cycle of $E$ has an entrance, and
2. $E$ has finite ancestry.

The restriction on $E$ allows us to use groupoid methods. Using a Drinen-Tomforde desingularization we can extend this to arbitrary graphs.
Theorem (C., ’13)

Let $E$ be a graph. Then $C^*(E)$ has continuous trace if and only if

1. no cycle of $E$ has an entrance, and
2. $E$ has finite ancestry.
Let $E$ be the graph

It can be shown that $C^*(E)$ has Hausdorff spectrum. While $E$ has no cycles, and hence no entrance to a cycle, it does not have finite ancestry. Thus $C^*(E)$ does not have continuous trace.
Part II: Stable graph algebras
Tensor products are common in $C^*$-algebras. Often you form from a $C^*$-algebra $A$ its stabilization $A \otimes K$, where $K$ is the $C^*$-algebra of compact operators on an infinite dimensional Hilbert space.
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**Definition**

A $C^*$-algebra $A$ is **stable** if it is isomorphic to $A \otimes \mathcal{K}$. 
The algebra $\mathcal{K}$ is stable because $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$. 
Example

The algebra $\mathcal{K}$ is stable because $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$.

Example

Any stable $C^*$-algebra is non-commutative and non-unital, so we get a wealth of non-stable $C^*$-algebras: $C_0(X), B(H), \mathcal{T}, O_2$, and others.
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**Lemma**

*Let $A$ be a stable $C^*$-algebra. Then $A$ has no tracial states.*
Stability

There are two properties of stable $C^*$-algebras that we will use over and over. A *tracial state* on a $C^*$-algebra is a positive linear functional $\phi$ of norm 1 such that $\phi(xy) = \phi(yx)$ for all $x, y \in A$.

**Lemma**

*Let $A$ be a stable $C^*$-algebra. Then $A$ has no tracial states.*

If $I$ is a two-sided closed ideal in a $C^*$-algebra then there is a quotient $C^*$-algebra $A/I$ and a canonical homomorphism $q : A \to A/I$.

**Lemma**

*Let $A$ be a stable $C^*$-algebra. Then $A$ has no nonzero unital quotients.*
Question
What conditions must a graph $E$ satisfy in order for $C^*(E)$ to be stable?
Stability

Discussing stability of graph algebras requires some new graph theory terminology.

Definition

A graph trace on a directed graph $E$ is a function $g : E^0 \to [0, \infty)$ satisfying

1. $g(v) \geq \sum_{r(e)=v} g(s(e))$ for all $v$
2. $g(v) = \sum_{r(e)=v} g(s(e))$ if $0 < |r^{-1}(v)| < \infty$

A graph trace is bounded if its norm $\sum_{v \in E^0} g(v)$ is finite. The (possibly empty) set of graph traces on $E$ with norm 1 is denoted by $T(E)$. 

Dan Crytser  
Traces on graph algebras
Graph traces lift to tracial states.

**Theorem (Tomforde ’03)**

If \( g \in T(E) \) then there is a tracial state \( \tau_g \) on \( C^*(E) \) such that \( \tau_g(p_v) = g(v) \).
Graph traces lift to tracial states.

**Theorem (Tomforde ’03)**

If $g \in T(E)$ then there is a tracial state $\tau_g$ on $C^*(E)$ such that $\tau_g(p_v) = g(v)$.

Stable $C^*$-algebras possess no tracial states. This shows that a graph with bounded graph traces cannot yield a stable $C^*$-algebra.
Left finite vertices

Definition

If \( v, w \in E^0 \), then we say that \( w \leq v \) if there is a directed path from \( v \) to \( w \). We say that \( v \) is \textit{left finite} if

\[
L(v) = \{ w \in E^0 : w \leq v \}
\]

is finite.

The following lemma tells us why we care about left finite vertices. Recall that a \textit{singular vertex} receives either zero edges or infinitely many edges.
Left finite vertices

**Definition**

If $v, w \in E^0$, then we say that $w \leq v$ if there is a directed path from $v$ to $w$. We say that $v$ is *left finite* if

$$L(v) = \{ w \in E^0 : w \leq v \}$$

is finite.

The following lemma tells us why we care about left finite vertices. Recall that a *singular vertex* receives either zero edges or infinitely many edges.

**Lemma**

*If $E$ has a left-finite vertex which lies on a cycle or is singular, then $C^*(E)$ has a nonzero unital quotient.*
Definition

Let $p, q$ be projections. We say that $p$ is subequivalent to $q$ if there exists an element $x$ such that $x^*x = p$ and $xx^* \leq q$.

Usually we will be comparing different projections of the form $p = \sum_{v \in V} p_v$ for some finite subset $V \subset E^0$. 

Projection comparison

Graph algebras
Continuous-trace graph algebras
Stable graph algebras

Stability
Stability of graph algebras

Dan Crytser
Traces on graph algebras
The following abridged theorem generalizes previous work of Hjelmborg [3].

**Theorem (Tomforde ’04)**

Let $E$ be a directed graph. Then the following are equivalent:

1. $C^*(E)$ is stable.
2. $C^*(E)$ has no tracial states and no nonzero unital quotients.
3. $E$ has no left finite cycles and no nonzero bounded graph traces.
4. For any $v \in E^0$ and any subset $F \subset E^0$, there exists $W \subset E^0 \setminus F$ such that $p_v \preceq \sum_{w \in W} p_w$. 
Proof

One part of the theorem needs reproving: the implication from (4) to (5).

4. $E$ has no left finite cycles, no left finite singular vertices, and no nonzero bounded graph traces.

5. For any $v \in E^0$ and any subset $F \subset E^0$, there exists $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$. 

The implication (4) implies (5) is the hardest to prove. The proof in the literature has a gap in it. I found a proof of this implication that seems novel and is "low-tech."
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Idea of proof: Show that if we cannot construct the comparison by using the “obvious” strategy, then the graph must carry a bounded graph trace.

First, let’s take a look at what this “obvious” strategy might be.

Dan Crytser
Proof

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**Idea of proof:** Show that if we cannot construct the comparison by using the “obvious” strategy, then the graph must carry a bounded graph trace. First, let’s take a look at what this “obvious” strategy might be.
Comparison of range and source

For any directed path $\lambda = e_1 e_2 \ldots e_n$ in a directed graph $E$, we have a partial isometry $s_\lambda = s_{e_1} s_{e_2} \ldots s_{e_n}$. The partial isometry $s_\lambda$ gives a subequivalence between $p_s(\lambda)$ and $p_r(\lambda)$, as $s_\lambda^* s_\lambda = p_s(\lambda)$ and $s_\lambda s_\lambda^* \leq p_r(\lambda)$.

**Lemma**

Suppose that $v$ is a left infinite vertex and $F \subset E^0$ is a finite set. Then there exists finite $W \subset E^0 \setminus F$ such that $p_v \precsim \sum_{w \in W} p_w$.

This allows us to restrict our attention to left finite vertices when we are constructing graph traces later on.
If $p_v$ is a vertex projection and $F \subset E^0$ then a cover for $\nu$ that avoids $F$ is a set of vertices $W$ with $p_v \preccurlyeq \sum_{w \in W} p_w$ and $W \cap F = \emptyset$. 
Example

Let $E$ be the graph

\[ v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \ldots \]

Notice that this graph does not carry a nonzero bounded graph trace.
Example

Let $E$ be the graph

We can cover vertex $v_0$ and avoid any finite $F = \{v_0, v_1, \ldots, v_n\}$. For

$$p_v = s_{e_1}s_{e_1}^* \sim s_{e_1}^*s_{e_1} = p_{v_1} = s_{e_2}s_{e_2}^* \sim s_{e_2}^*s_{e_2} = p_{v_2} \sim \ldots \sim p_{v_{n+1}}.$$ 

Thus $p_{v_{n+1}}$ is a cover for $p_v$ and we can take $W = \{v_{n+1}\}$. 
Example

Let $E$ be the graph

\begin{center}
\begin{tikzpicture}
  \node[shape=circle,draw, inner sep=0pt, minimum size=5mm,fill] (A) at (0,0) {$v_0$};
  \node[shape=circle,draw, inner sep=0pt, minimum size=5mm,fill] (B) at (1,0) {$v_1$};
  \node[shape=circle,draw, inner sep=0pt, minimum size=5mm,fill] (C) at (2,0) {$v_2$};
  \node[shape=circle,draw, inner sep=0pt, minimum size=5mm,fill] (D) at (3,0) {$v_3$};
  \node[shape=circle,draw, inner sep=0pt, minimum size=5mm,fill] (E) at (4,0) {$v_4$};

  \draw (A) -- (B) node[pos=0.5, above] {$e_1$};
  \draw (B) -- (C) node[pos=0.5, above] {$e_2$};
  \draw (C) -- (D) node[pos=0.5, above] {$e_3$};
  \draw (D) -- (E) node[pos=0.5, above] {$e_4$};

\end{tikzpicture}
\end{center}

We can cover vertex $v_0$ and avoid any finite $F = \{v_0, v_1, \ldots, v_n\}$. For

\[ p_v = s_{e_1}s_{e_1}^* \sim s_{e_1}^*s_{e_1} = p_{v_1} = s_{e_2}s_{e_2}^* \sim s_{e_2}^*s_{e_2} = p_{v_2} \sim \ldots \sim p_{v_{n+1}}. \]

Thus $p_{v_{n+1}}$ is a cover for $p_v$ and we can take $W = \{v_{n+1}\}$. Notice that this graph does not carry a nonzero bounded graph trace.
Example

Let $E$ be the graph

\[ \begin{array}{c}
  v_0 & \xrightarrow{e_1} & v_1 & \xrightarrow{e_2} & v_2 & \xrightarrow{e_3} & v_3 & \xrightarrow{e_4} & \ldots \\
  \xleftarrow{e_1'} & & \xleftarrow{e_2'} & & \xleftarrow{e_3'} & & \xleftarrow{e_4'} & & \\
\end{array} \]
Example

Let $E$ be the graph

\[ \begin{array}{cccccccc}
  v_0 & e_1 & v_1 & e_2 & v_2 & e_3 & v_3 & e_4 & \ldots \\
  e_1' & & v_1 & e_2' & v_2 & e_3' & v_3 & e_4' & \\
\end{array} \]

**Claim:** we can’t cover $v_0$ and avoid $F = \{v_0\}$. We have $p_{v_0} = s_{e_1} s_{e_1}^* + s_{e_1'} s_{e_1'}^*$. Then $s_{e_1} s_{e_1}^* \sim s_{e_1'} s_{e_1'} = p_{v_1}$ and likewise for $s_{e_1'} s_{e_1}^*$. However we can’t write $p_v \preceq p_{v_1} + p_{v_1}$ because the sum is not a projection. So we cover one range projection and split the other. But this lands us exactly where we started. This process goes on forever.
Example

Let $E$ be the graph

```
<table>
<thead>
<tr>
<th>e_1</th>
<th>e_2</th>
<th>e_3</th>
<th>e_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>v_0</td>
<td>v_1</td>
<td>v_2</td>
<td>v_3</td>
</tr>
<tr>
<td>e'_1</td>
<td>e'_2</td>
<td>e'_3</td>
<td>e'_4</td>
</tr>
</tbody>
</table>
```

Claim: we can’t cover $v_0$ and avoid $F = \{v_0\}$. We have $p_{v_0} = s e_1 s^{*} e_1 + s e'_1 s^{*} e'_1$. Then $s e_1 s^{*} e_1 \sim s^{*} e_1 s e_1 = p_{v_1}$ and likewise for $s e'_1 s^{*} e'_1$. However we can’t write $p_{v} \preccurlyeq p_{v_1} + p_{v_1}$ because the sum is not a projection. So we cover one range projection and split the other. But this lands us exactly where we started. This process goes on forever. Note that this graph carries a bounded trace with $g(v_i) = \frac{1}{2^{i+1}}$. 
Now let’s sketch the proof of

4. \( E \) has no left finite cycles, no left finite singular vertices, and no nonzero bounded graph traces.

5. For any \( v \in E^0 \) and any subset \( F \subset E^0 \), there exists \( W \subset E^0 \setminus F \) such that \( p_v \lesssim \sum_{w \in W} p_w \).

Suppose that \( v \) is a regular vertex of \( E \) and \( F \) is a finite subset of \( E^0 \) such that for all \( W \subset E^0 \), we have \( p_v \lesssim \sum_{w \in W} p_w \).
Assume that all $N_1$ edges entering $v$ have common source

Then $p_v = \sum_{r(e)=v} s_e s_e^*$. Let $d_1$ be the number of paths $\lambda_1, \ldots, \lambda_{d_1}$ which start at $v_1$ and terminate at a vertex not in $F$. If $d_1 \geq N_1$, then we can write $p_v \lesssim \sum_{w \in r(\{\lambda_i\})} p_w$.

Thus we must have $d_1 < N_1$, or equivalently $\frac{d_1}{N_1} < 1$. 
Now assume we couldn’t find a comparison using edges going into \( v \).

Let \( N_2 \) be the number of edges from \( v_2 \) to \( v_1 \), and let \( d_2 \) be the number of paths which start at \( v_2 \), don’t include the \( N_1 \) edges from \( v_1 \) to \( v \), and don’t terminate in \( F \). If \( d_2 \geq N_2(N_1 - d_1) \), then we can construct the comparison. So we must have that \( d_2 < N_2(N_1 - d_1) \), or equivalently that \( \frac{d_1}{N_1} + \frac{d_2}{N_1N_2} < 1 \).
Definition of the graph trace

Inductively we find a chain of vertices $v, v_1, \ldots$ with $N_i$ vertices from $v_i$ to $v_{i-1}$, and $d_i$ paths out of $v_i$ which do not terminate at a vertex in $F$. The nice thing about this chain is

$$\sum_{i=1}^{\infty} \frac{d_i}{N_1 \ldots N_i} < 1$$
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If $w \in E^0$, define

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g(w) = \sum_{i=1}^{\infty} \frac{|\{\text{nice paths } w \leftarrow v_i\}|}{N_1 \ldots N_i}.
$$

You can check that this is a graph trace.
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You can check that this is a graph trace. Bounded? Need to worry about the paths which terminate in the finite set $F$, but they just multiply the trace norm by a finite constant.
Thus we have seen that the failure of comparison within a $C^*$-algebra associated to a graph with left infinite cycles and singular vertices yields a nonzero graph trace on the graph, and hence a tracial state on the $C^*$-algebra. This seals the gap in the theorem on stability for graph algebras.
Stable \( k \)-graph algebras

Directed graphs can be generalized to more combinatorially rich objects called \( k \)-graphs.

**Definition**

A \( k \)-graph \( \Lambda \) is a category equipped with a degree functor \( d : \Lambda \rightarrow \mathbb{N}^k \) which satisfies the factorization property: if \( d(\lambda) = m + n \) for some \( m, n \in \mathbb{N}^k \), then there is a unique factorization of \( \lambda \) as \( \lambda = \mu \nu \) with \( d(\mu) = m \) and \( d(\nu) = n \). The objects of \( \Lambda \) are precisely \( d^{-1}(0) = \Lambda^0 \). In general if \( n \in \mathbb{N}^k \), then \( \Lambda^n \) denotes \( d^{-1}(n) \).
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A $k$-graph $\Lambda$ is a category equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the *factorization property*: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there is a unique factorization of $\lambda$ as $\lambda = \mu \nu$ with $d(\mu) = m$ and $d(\nu) = n$. The objects of $\Lambda$ are precisely $d^{-1}(0) = \Lambda^0$. In general if $n \in \mathbb{N}^k$, then $\Lambda^n$ denotes $d^{-1}(n)$.

We can assign a $C^*$-algebra to a well-behaved $k$-graph in a manner very similar to the definition of graph algebras. It then becomes interesting to ask which $k$-graphs yield stable $C^*$-algebras.
Stable $k$-graph algebras

I wanted to look at a class of $k$-graphs which is amenable to the construction of $k$-graph traces developed by Evans, Rennie and Sims.

**Definition**

A $k$-graph is *balanced* if for any basis elements $e_i, e_k \in \mathbb{N}^k$, we have $|v \Lambda^{e_i} w| = |v \Lambda^{e_k} w|$.
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Theorem (work in progress)

Let $\Lambda$ be a row-finite balanced $k$-graph with no sources. Then the following are equivalent.

1. $C^*(\Lambda)$ is stable;
2. $C^*(\Lambda)$ has no tracial states and no nonzero unital quotients;
3. no left finite $v \in \Lambda^0$ lies on a cycle and $\Lambda$ has no nonzero bounded $k$-graph traces.
The notion of a balanced $k$-graph above includes nice examples of $k$-graphs, but it’s fairly restrictive.
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The notion of a balanced $k$-graph above includes nice examples of $k$-graphs, but it’s fairly restrictive. I think that I can extend it to vertex-balanced $k$-graphs: $k$-graphs in which every vertex receives the same number of edges of degree $e_i$ for every basis element in $\mathbb{N}^k$. This class includes more interesting examples of $k$-graphs than the balanced class. The combinatorics involved in constructing the $k$-graph traces under failure of comparison becomes more complicated.
Partial bibliography


Thank you!