Math 22: Final Exam
November 16, 2012, 3pm-6pm

Your name (please print): ____________________________

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. Unless otherwise stated, you must justify all of your answers to receive credit - please write in complete sentences in a paragraph structure. You may not give or receive any help on this exam and all questions should be directed to Professor Pauls.

You have 3 hours to work on all 9 problems. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.
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(1) (10 points) Complete the following definitions - remember, state definitions of the terms, not properties of the terms. To get credit, your answers must make sense as English sentences.

(a) A set of vectors is linearly independent if . . .

(b) A map \( T: V \rightarrow W \) is a linear transformation of vectors spaces if . . .

(c) A matrix \( A \) is invertible if . . .
(d) Let $B$ be an $n \times n$ matrix. Then, a vector $\vec{v}$ is an eigenvector of $A$ if . . .

(e) A set of vectors $\mathfrak{B} = \{\vec{v}_1, \ldots, \vec{v}_k\} \subset V$ is a basis for the vector space $V$ if . . .

(f) A matrix $C$ is an orthogonal matrix if . . .
(g) A a Markov chain is . . .

(h) Let $D$ be a square matrix. Then, the characteristic polynomial of $D$ is . . .

(i) The rank of a matrix is . . .
(j) The least squares solution to the matrix equation $A\vec{x} = \vec{b}$ is $\ldots$
(2) (35 points total, 5 points each) For each question, explain your process and write clearly. All answers must be fully justified, especially answers to yes or no questions.

(a) Let \( A_1 = \begin{pmatrix} 1 & 2 & -4 & -4 \\ 2 & 4 & 0 & 0 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 3 & 6 \end{pmatrix} \) and \( \vec{b} = \begin{pmatrix} 5 \\ 2 \\ 5 \\ 5 \end{pmatrix} \). Find all solutions to the matrix equation \( A_1 \vec{x} = \vec{b} \) or show that no solutions exist.
(b) Let $A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{pmatrix}$. Show that the columns of $A_2$ are either linearly dependent or linearly independent. What does this say about the dimension of $\text{Col } A$? Does this imply anything about the dimension of $\text{Nul } A$? If so, what and why?
(c) Let \( A_3 = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{pmatrix} \). Find a basis for \( \text{Nul} \ A_3 \). What is the rank of \( A_3 \)? Is \( A_3 \) invertible?
(d) Let $A_4 = \begin{pmatrix} 3 & -1 & 5 \\ 2 & 1 & 3 \\ 0 & -5 & 1 \end{pmatrix}$. Find a basis for $\text{Row } A_4$. What is the rank of $A_4$?

What is the dimension of $\text{Nul } A$?
(e) Let $A_5 = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & 2 \end{pmatrix}$. Find a basis for $\text{Col} A_5$. What is the rank of $A_5$?
(f) Let $A_6 = \begin{pmatrix} 13 & -5 & 2 \\ -5 & 13 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Compute the determinant of $A_6$. Is $A$ invertible?
(g) Let $A_7 = \begin{pmatrix} 13 & -5 & 1 \\ -6 & 10 & 3 \\ -5 & -2 & 3 \end{pmatrix}$. The eigenvalues of this matrix are 1, 2 and 3. Find all the eigenvectors of $A_7$. Is $A_7$ diagonalizable? If so, give the diagonalization.
(3) (10 points) Let $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Compute the reduced singular value decomposition of $B$. Does $B$ have a trivial or non-trivial null space? What is the rank of $B$?
(b) Find the pseudo-inverse of $B$. 
(4) (10 points) Let $Q$ be an $n \times n$ orthogonal matrix and $A$ an $n \times m$ matrix. Show that $A$ and $QA$ have the same singular values.
(5) (10 points) Let $C$ be a $3 \times 3$ symmetric matrix with orthogonal diagonalization given by $C = PDP^{-1}$ where the columns of $P$ are $\{\vec{p}_1, \ldots, \vec{p}_n\}$ and the nonzero entries of the matrix $D$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ where $r < n$. Let $\mathfrak{B}$ denote the basis of eigenvectors of $C$.

(a) What is the change of basis matrix from the standard basis to $\mathfrak{B}$? What is the change of basis matrix from $\mathfrak{B}$ to the standard basis (do not just state this as an inverse of another matrix)?
(b) What is $[C]_g$? Justify your answer.
(6) (5 points) Let
\[
D = \begin{pmatrix}
1 & 2 & 2 \\
-1 & 1 & 2 \\
-1 & 0 & 1 \\
1 & 1 & 2
\end{pmatrix}
\]

\(D\) has a QR decomposition given by
\[
D = QR = \begin{pmatrix}
\frac{1}{2} & \frac{3\sqrt{5}}{10} & -\frac{\sqrt{5}}{6} \\
-\frac{1}{2} & \frac{3\sqrt{5}}{10} & 0 \\
-\frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{6} \\
\frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{3}
\end{pmatrix} \begin{pmatrix}
2 & 1 & \frac{1}{2} \\
\sqrt{5} & 0 & \frac{3\sqrt{5}}{2} \\
0 & \frac{\sqrt{5}}{2} & 0
\end{pmatrix}
\]

Using the QR factorization, find the least squares solution to \(A\vec{x} = \vec{b}\) where
\[
\vec{b} = \begin{pmatrix}
2 \\
-3 \\
-2 \\
0
\end{pmatrix}
\]
(7) (10 points) Describe and explain the Gram-Schmidt algorithm.
Consider the following data series:

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<tr>
<td>y</td>
<td>0</td>
<td>2</td>
<td>1</td>
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Suppose we wish to construct a general linear model of the form $y = \beta_1 x + \beta_2 x^3$. What is are design matrix, observation vector and parameter vector for this model? Write down the normal equations for this model but do not solve them.
(9) (10 points) Let $A$ be an $m \times n$ matrix. Show that $Nul A$ is a subspace of $\mathbb{R}^n$ and that $Row A$ is its orthogonal complement.
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