Math 22: Exam 2

Solutions

October 30, 2012, 6pm-8pm

Your name (please print): ______________________________

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted.** Unless otherwise stated, you must justify all of your answers to receive credit - please write in complete sentences in a paragraph structure. You may not give or receive any help on this exam and all questions should be directed to Professor Pauls.

You have **2 hours** to work on all **8** problems. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Warning: Due to having different syllabi, the following topics are not exam questions:

- the row space of a matrix
- the change of basis matrix
- the vector space of polynomials
- any proof-based question (especially question 8)

Be careful, sometimes parts of a question would still be exam material even if others aren't (e.g. 3 (a) vs 3 (c)).

Non-exam questions are denoted by NEX

(1) (10 points, 5 each) Let A be an $n \times m$ matrix.

(a) Define the null space, the column space and the row space of A.

The null space of A, $Nul(A)$, is a subspace of $\mathbb{R}^m$ whose vectors satisfy the equation $Ax=0$

The column space of A, $Col(A)$, is a subspace of $\mathbb{R}^n$ whose vectors are the linear combinations of the columns of A.

NEX : The row space of A, $Row(A)$, is a subspace of $\mathbb{R}^m$ whose vectors are the linear combinations of the rows of A.

(b) Identify the vector space of which *Row A* is a subset. Show that *Row A* is a subspace of that vector space.

NGT: As in (a), because A is $m \times n$ the row space is a subspace of $\mathbb{R}^n$. If we denote the rows by $r_1 \dots r_m$ then

- $0 \in Row(A)$ b/c $0 = 0r_1 + \dots + 0r_m$
- If $u, v \in Row(A)$, $u+v \in Row(A)$ and $cu \in Row(A)$ because $Row(A) = Span\{r_1 \dots r_m\}$.

(2) (15 points) Which of the following matrices are invertible? For the 2×2 matrices, if they are invertible, find their inverses. In each case justify your answer!

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 4 & -2 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

Note: There are many ways to check for invertibility. I use the determinant, but any other method would be been acceptable

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2 \Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

$$\det\begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 4 & -2 & 7 \end{pmatrix} = 1 \cdot \det\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} = 1 \cdot (4-4) = 0 \Rightarrow \text{not invertible}$$

$$\det\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = 6 - 6 = 0 \Rightarrow \text{not invertible}$$

(3) (20 points total) The matrix A given by

$$\begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}$$

has reduced echelon form given by

$$\begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is actually not reduced, need to divide second row by the pivot.

(a) (2 points) Find a basis, $\mathfrak{B}$, for *Row A*.

NEX

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 5 \\ -6 \end{pmatrix} \right\}$$

(just read off the nonzero rows in the echelon form)

(b) (3 points) If we label your basis from part a) as $\mathfrak{B} = \{\vec{b}_1, \dots, \vec{b}_k\}$, is $\{A\vec{b}_1, \dots, A\vec{b}_k\}$ a basis for $Col\ A$? Justify your answer!

Yes

Since $\dim(Row(A)) = \dim(Col(A)) = rank(A)$,

and $Span\{Ab_1, \dots, Ab_k\}$ is a subspace of $Col(A)$

(because each Ab_i is a linear combination of the columns of A), then $\{Ab_1, \dots, Ab_k\}$ is a basis for $Col(A)$.

(c) (3 points) Find a basis, $\mathfrak{N}$, for $Nul\ A$.

$$\begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 - 5x_4 \\ \frac{5}{2}x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

$$Nul(A) = Span\left\{ \begin{pmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(d) (3 points) Show that the union of the vectors in $\mathfrak{N}$ and $\mathfrak{B}$ is a basis for $\mathbb{R}^4$.

$$\begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & -2 & 5/2 & -3 \\ -1 & 5 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & -2 & 5/2 & -3 \\ 0 & 5 & 2 & -5 \\ 0 & 6 & -5 & 26 \end{pmatrix} \sim$$

$$\underset{\substack{R_2 = R_4 - R_3 \\ R_3 = R_2}}{\sim} \begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & 1 & -7 & 31 \\ 0 & -2 & 5/2 & -3 \\ 0 & 6 & -5 & 26 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & 1 & -7 & 31 \\ 0 & 0 & -23/2 & 59 \\ 0 & 0 & 37 & -160 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & 1 & -7 & 31 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ LI $\Rightarrow$ basis for $\mathbb{R}^4$.

Note: the system you get is a bit too complicated in my opinion. It is possible that the intended way of solving it was to argue that Row(A), Nul(A) are always independent spaces, and since $\text{rank}(A) + \dim(\text{Nul}(A)) = 4$ you always get that $\mathfrak{N} \cup \mathfrak{B}$ is a basis for $\mathbb{R}^4$. However, this could've been a doable exam question using row-reduction

(e) (1 points) Give a change of basis matrix from the standard basis to the basis in the previous part (if the answer is the inverse of a matrix, you need not compute the inverse).

NEX If the standard basis of $\mathbb{R}^4$ is $\mathcal{E} = \{e_1, \dots, e_4\}$,

then the change of basis matrix satisfies $\underset{(\eta \cup \mathcal{B}) \leftarrow \mathcal{E}}{P} = \left(\underset{\mathcal{E} \leftarrow (\eta \cup \mathcal{B})}{P} \right)^{-1}$

and $\underset{\mathcal{E} \leftarrow \eta \cup \mathcal{B}}{P} = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} 0 \\ -2 \\ 5 \\ 6 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & -2 & 5/2 & -3 \\ -1 & 5 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{pmatrix}$

So the answer is $\begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & -2 & 5/2 & -3 \\ -1 & 5 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{pmatrix}^{-1}$ (which you do not need to compute

(4) (10 points total) Consider the matrix A given by

$$\begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}$$

A has a double eigenvalue of 3.

(a) (5 points) Find all of the eigenvectors associated with the eigenvalue 3.

$$A - 3I = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

eigenvectors are $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

(b) (5 points) Is A diagonalizable? If not, why? If so, show the diagonalization.

Note: The question seems to be lacking the information that $\lambda = 8$ is an eigenvalue as well. It's also unclear what "show the diagonalization" means (D? P & D?).

- Eigenvector for $\lambda = 8$

$$A - 8I = \begin{pmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & 3 \\ 0 & 30 & 15 \\ 0 & -10 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow X = x_3 \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

- diagonalization form

$$P = \begin{pmatrix} -2 & -3 & 1/2 \\ 1 & 0 & -1/2 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(5) (10 points total) Let A be

This should be -1

$$\begin{pmatrix} 2 & -2 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{pmatrix}$$

and

$$\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

Solve $A\vec{x} = \vec{b}$ using the LU factorization for A given by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

NEX In our midterm (or rather, final) I'd just ask you to solve the system.

Using LU we're solving $\begin{cases} Ux = y \\ Ly = b \end{cases}$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 4 & -1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & -3 & 4 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 & -3 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} -1/2 \\ 2 \\ 2 \end{pmatrix}$$

Note: The LU factorization is wrong (or rather, the A matrix is wrong). The product of L and U gives

$$\begin{pmatrix} 2 & -1 & 2 \\ -4 & -1 & 0 \\ 8 & -1 & 5 \end{pmatrix}$$

(6) (15 points total) Let P be a regular Markov chain given by

$$\begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix}$$

(a) (5 points) Find all the eigenvalues and associated eigenvectors of P.

$$P - \lambda I = \begin{pmatrix} 0.6-\lambda & 0.5 \\ 0.4 & 0.5-\lambda \end{pmatrix}$$

$$\det(P - \lambda I) = (0.6-\lambda)(0.5-\lambda) - (0.5)(0.4) =$$

$$= 0.30 - 1.1\lambda + \lambda^2 - 0.20$$

$$= \lambda^2 - 1.1\lambda + 0.1$$

$$= (\lambda - 1)(\lambda - 0.1)$$

$$P - I = \begin{pmatrix} -0.4 & 0.5 \\ 0.4 & -0.5 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/4 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 5/4 \\ 1 \end{pmatrix} \left[\text{or } \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right]$$

$$P - 0.1I = \begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \left(\text{or } \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

(b) (10 points) What does $P^k\vec{x}$ converge to as k approaches infinity? Justify your answer completely!

Possible bonus question

To know what $P^k x$ converges to

we write $x = c_1 v_1 + c_2 v_2$ (picking $v_1 = \begin{pmatrix} 5/4 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$)

We know we can do this because the vectors are L.I. and thus form a basis for $\mathbb{R}^2$.

But then $Px = P(c_1 v_1 + c_2 v_2) = c_1 (Pv_1) + c_2 (Pv_2) =$

$= c_1 \cdot 1 \cdot v_1 + c_2 \cdot (0.1) \cdot v_2$

So $P^k x = c_1 v_1 + c_2 \cdot (0.1)^k \cdot v_2$, and as $k \to \infty$

$P^k x \to c_1 v_1$.

(7) (13 points total) Consider the linear transformation $T : P_2 \to \mathbb{R}^3$ given by

$$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1, 3a_1 + 2a_2, 4a_2)$$

(a) (4 points) Using the basis $\mathfrak{B} = \{1, x, x^2\}$ for P_2 and the standard basis $\mathscr{E}$ for $\mathbb{R}^3$, give a matrix representation, A, for T with respect to these two bases.

NEX. Since we did not cover polynomials as a vector space (or matrix representations) this would not be an exam question.

Using the two bases we can rewrite the expression as

$$A \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_0 + a_1 \\ 3a_1 + 2a_2 \\ 4a_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\left[[a_0 + a_1x + a_2x^2]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \right]$$

(b) (2 points) What is the rank of T? Justify your answer!

Since A has three pivots $\operatorname{rank}(T) = 3$

(c) (5 points) Find the eigenvalues and eigenvectors of A. If possible, give a diagonalization of A.

Because A is triangular, we can read its eigenvalues from the main diagonal

$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 4$

Since these are all distinct A will be diagonalizable.

A diagonalization of A will be $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(Note: It's again unclear if P is requested or not. Since the question is only worth 5 points I don't think that's the case. I will always specify if I want P or not)

(d) (2 points) Is A invertible? Justify your answer.

Yes, zero is not an eigenvalue, so A is invertible.

Or, Yes b/c $\det(A) = \underset{1 \cdot 3 \cdot 4}{\overset{\shortparallel}{12}} \neq 0$

(8) (15 points total)

(a) (5 points) Show that if A and B are similar matrices then $det\ A = det\ B$.

Possible bonus question

Because $\det(AB) = \det(A)\det(B)$,
and similar matrices satisfy $A = PBP^{-1}$,

$\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1}) =$
$= \det(B)\det(P)\det(P^{-1}) = \det(B)\det(PP^{-1}) =$
$= \det(B)\underbrace{\det(I)}_{\shortparallel\, 1} = \det(B)$

(b) (5 points) Show that similar matrices have the same eigenvalues.

NEX This is the proof of Thm 4, p277 of the book.

Basically it's because if $A = P^{-1}BP$ then $A - \lambda I_n = P^{-1}(B - \lambda I_n)P$ and part (a) of the question.

(c) (5 points) Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

NEX Eigenvectors corresponding to distinct eigenvalues are LI. If there were $n+1$ distinct eigenvalues, $\mathbb{R}^n$ would have $n+1$ LI vectors, so

$\underbrace{\dim(\mathbb{R}^n)}_{= n} \geq n+1$, which is a contradiction.