Crash Course in Partial Differentiation

The difference between ordinary differentiation (denoted by $\frac{d}{dx}$) and partial differentiation (denoted $\frac{\partial}{\partial x}$) lies in the idea of dependency of variables.

When taking an ordinary derivative (say $\frac{d}{dx}$) of a function of one or more variables, it must be assumed that all the variables depend on x. For example, consider the function

$$f(x,y) = (\ln y)\sin x + y^2.$$

To compute $\frac{df}{dx}$ we must treat the variable y as a function of x and use the chain-rule (and sometimes as here, others such as the product rule). Thus

$$\frac{df}{dx} = (\ln y)\cos x + \sin x \frac{1}{y}\frac{dy}{dx} + 2y\frac{dy}{dx}$$

or if computing $\frac{df}{dy}$ we must treat x as a function of y

$$\frac{df}{dy} = \frac{1}{y}\sin x + (\ln y)\cos x\frac{dx}{dy} + 2y.$$

For partial derivatives it is instead assumed that all the variables are independent. Thus when applying $\frac{\partial}{\partial x}$ to a function of several variables, you can assume that all variables apart from x are constants. With the above example we get

$$\frac{\partial f}{\partial x} = (\ln y)\cos x, \qquad \frac{\partial f}{\partial y} = \frac{1}{y}\sin x + 2y.$$

There is a close connection between the two ideas and the chain rule. Partial derivatives are an important component of the computation of an ordinary derivative. For a function of 3 variables $f(x,y,z)$ the derivative $\frac{df}{dx}$ can be computed by the formula

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}.$$

Similar formulae hold for functions of any number of variables.