Resonance

Resonance is a very general phenomenon that happens so often and in so many systems that we sometimes ignore its existence. In the most general sense, resonance represents that the energy in a system exchanges from one form into another at a particular rate.

Some examples of resonance:

•Bridges or buildings resonate when excited by external forces (cars or passengers passing by).
•Circuits resonate when electric energy stored in capacitors and magnetic energy stored in inductors exchange back and forth.
•Lasers are "resonant cavities" where photons and electrons exchange energies at particular frequencies (lasing frequencies).
•Any musical instrument resonates at frequencies defined by its string length or cavity size. (exchange of the kinetic and potential energy like an oscillating spring)
•Pendulum resonates at a precise frequency, the exchange rate between potential energy and kinetic energy.
•Examples go on and on.

Two quantities characterize the effect of resonance:

1. Resonance frequency.
2. Resonance bandwidth, or the ratio between the bandwidth and the resonance frequency as defined as the Q-factor (quality factor).

A resonator has low quality if its energy is lost quickly during the energy exchange process. Losses may be caused by "friction" in mechanical systems, and "resistor" in electronic systems.

For the parallel *RLC* circuit

the admittance loading the current source is $\mathbf{Y} = \mathbf{I}/\mathbf{V}$,

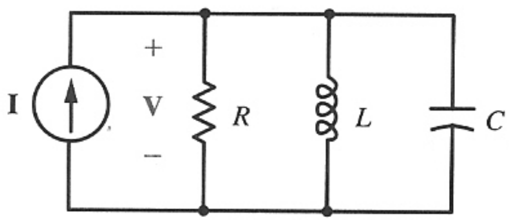


Fig. 5.11 Parallel *RLC* circuit.

$$\mathbf{Y} = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}_C = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

From this expression, we can see that there is some frequency ω_r for which the imaginary part of $\mathbf{Y}$ will be zero, that is,

$$\omega_r C - \frac{1}{\omega_r L} = 0 \quad \Rightarrow \quad \omega_r = \frac{1}{\sqrt{LC}}$$

$f_r = \omega_r/2\pi$: resonant frequency

Bandwidth: separation of the "half-power" frequencies, or sometimes we call FWHM (Full-Width-Half-Maximum).

$$BW = \omega_2 - \omega_1$$

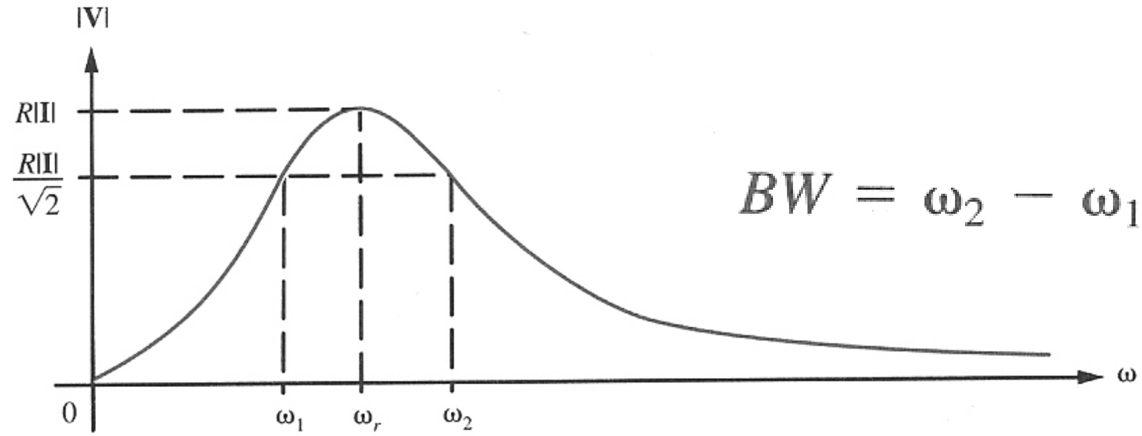


Fig. 5.12 Voltage magnitude versus frequency.

Quality factor:

$$Q = 2\pi\left(\frac{\text{maximum energy stored}}{\text{total energy lost in a period}}\right)$$

High quality factor→ low loss→narrow bandwidth or sharp resonant curve.

For electronic circuit: $Q = \dfrac{2\pi[w_C(t) + w_L(t)]_{\max}}{P_R T}$

Then the energy stored in the capacitor $w_C(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2}C(RI\cos\omega_r t)^2 = \frac{1}{2}CR^2I^2\cos^2\omega_r t$

At resonance, the energy stored in the inductor

$$w_L(t) = \frac{1}{2}Li_L^2(t) = \frac{1}{2}\left(\frac{R^2I^2}{\omega_r^2 L}\right)\sin^2\omega_r t = \frac{1}{2}CR^2I^2\sin^2\omega_r t$$

Total energy stored:

$$w_C(t) + w_L(t) = \frac{1}{2}CR^2I^2(\cos^2 \omega_r t + \sin^2 \omega_r t) = \frac{1}{2}CR^2I^2$$

Note that the total energy stored at resonance is time independent.

Energy lost per time period: $P_R T = \frac{1}{2}RI^2T = \frac{1}{2}RI^2\left(\frac{2\pi}{\omega_r}\right) = \frac{\pi RI^2}{\omega_r}$

Quality factor for the ***PARALLEL*** RLC circuit: $Q = \dfrac{2\pi(\frac{1}{2}CR^2I^2)}{\pi RI^2/\omega_r} = \omega_r RC$

$$= \omega_r RC = \frac{R}{\omega_r L} = R\sqrt{\frac{C}{L}}$$

Relationship Between Q and Bandwidth

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) = \frac{1}{R} + j\left(\frac{\omega C \omega_r R}{\omega_r R} - \frac{\omega_r R}{\omega L \omega_r R}\right)$$

$$= \frac{1}{R} + j\frac{1}{R}\left(\frac{\omega}{\omega_r}Q - \frac{\omega_r}{\omega}Q\right) = \frac{1}{R}\left[1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right]$$

At the half-power frequencies ω_1 and ω_2,

$$|\mathbf{V}| = \frac{R|\mathbf{I}|}{\sqrt{2}} = \frac{|\mathbf{I}|}{|\mathbf{Y}|} \quad \Rightarrow \quad |\mathbf{Y}| = \frac{\sqrt{2}}{R}$$

This occurs when $Q\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right) = \pm 1$

We have

$$\omega_2 = \frac{\omega_r}{2Q} + \omega_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \qquad \omega_1 = \frac{-\omega_r}{2Q} + \omega_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1}$$

$$BW = \omega_2 - \omega_1 = \frac{\omega_r}{Q}$$

Example 5.3

For the parallel RLC circuit shown in Fig. 5.11 on p. 276, suppose that $R = 10\ \text{k}\Omega$, $L = 50.7\ \mu\text{H}$, and $C = 500$ pF. Then the resonance frequency is

$$\omega_r = \frac{1}{\sqrt{LC}} = 6.28 \text{ Mrad/s} \qquad (1 \text{ MHz})$$

$$Q = R\sqrt{\frac{C}{L}} = 31.4$$

$$BW = \frac{1}{RC} = 200 \text{ krad/s} \qquad (31.8 \text{ kHz})$$

The lower half-power frequency is

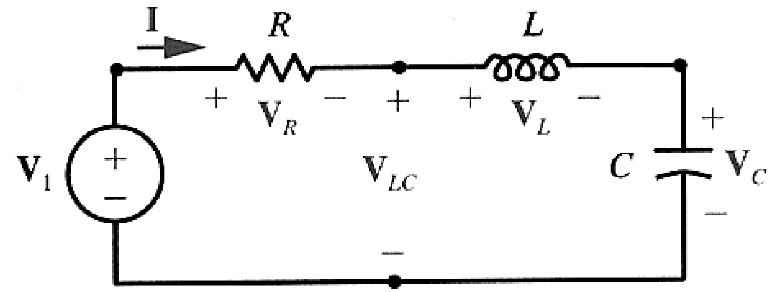
$$\omega_1 = -\frac{\omega_r}{2Q} + \omega_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1} = 6.18 \text{ Mrad/s} \qquad (984 \text{ kHz})$$

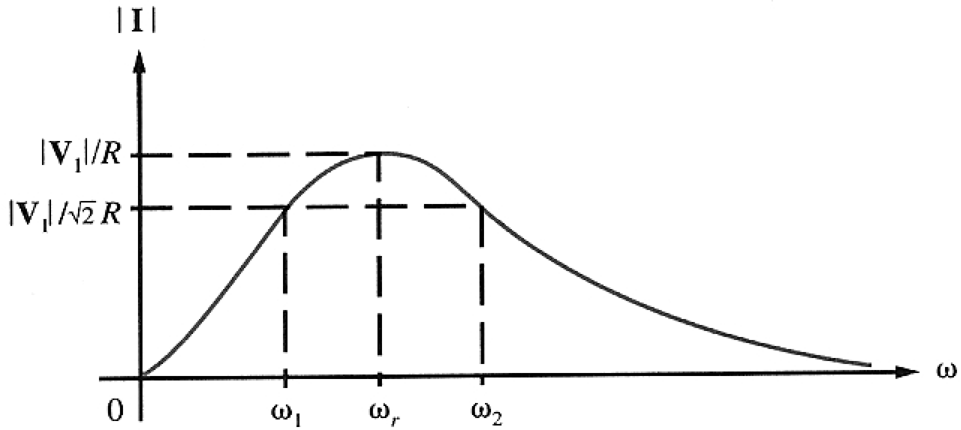
the upper half-power frequency is

$$\omega_2 = \frac{\omega_r}{2Q} + \omega_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1} = 6.38 \text{ Mrad/s} \qquad (1.016 \text{ MHz})$$

Series Resonance

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$





$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_r L}{R} = \frac{1}{\omega_r RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

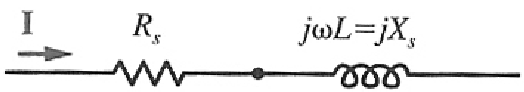
$$BW = \frac{\omega_r}{Q} = \frac{R}{L}$$

$$\omega_1 = -\frac{\omega_r}{2Q} + \omega_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1}$$

$$\omega_2 = \frac{\omega_r}{2Q} + \omega_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1}$$

Series and Parallel Reactances → **Q becomes frequency dependent**

Just as we defined the quality factor Q for a resonant circuit, we can define it for a reactance in series or parallel with a resistance.



$$w(t) = \tfrac{1}{2}Li^2(t) = \tfrac{1}{2}LI^2\cos^2\omega t$$

Fig. 5.14 Series inductive reactance.

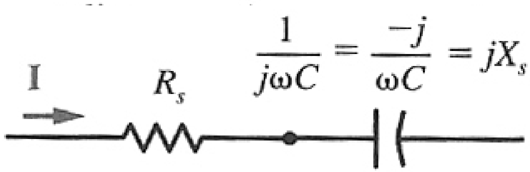
which has the maximum value $w_m = \frac{1}{2}LI^2$. Since $X_s = \omega L \Rightarrow L = X_s/\omega$, then

$$w_m = \frac{1}{2}\left(\frac{X_s}{\omega}\right)I^2$$

The energy lost per period is

$$P_sT = \left(\frac{1}{2}R_sI^2\right)\left(\frac{2\pi}{\omega}\right) = \frac{\pi R_sI^2}{\omega} \quad \Longrightarrow \quad Q = 2\pi\frac{\frac{1}{2}(X_s/\omega)I^2}{\pi R_sI^2/\omega} = \frac{X_s}{R_s}$$

For the case of a resistor in series with a capacitor C as shown in Fig. 5.15,



$$\Longrightarrow \quad Q = \frac{|X_s|}{R_s}$$

Fig. 5.15 Series capacitive reactance.

Capacitors with a parallel resistor:

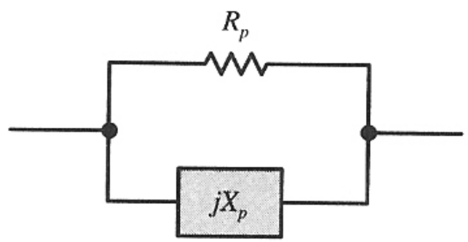


Fig. 5.16 Parallel reactance.

$$Q = \frac{R_p}{|X_p|}$$