Math 23 WORKSHEET: Resonance

Barnett
10/17/05

FACTS

Driven system $mu'' + \gamma u' + ku = \underbrace{F}_{\text{driving ampl.}} \cos \underbrace{\omega}_{\text{driving freq.}} t$

natural freq $\omega_0 = \sqrt{\frac{k}{m}}$

has steady-state response $U(t) = R\cos(\omega t + \delta)$

ampl. $R = \dfrac{F}{\Delta}$, with $\Delta^2(\omega) = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$

phase $\tan\delta = \dfrac{\gamma}{m}\dfrac{\omega}{\omega_0^2 - \omega^2}$.

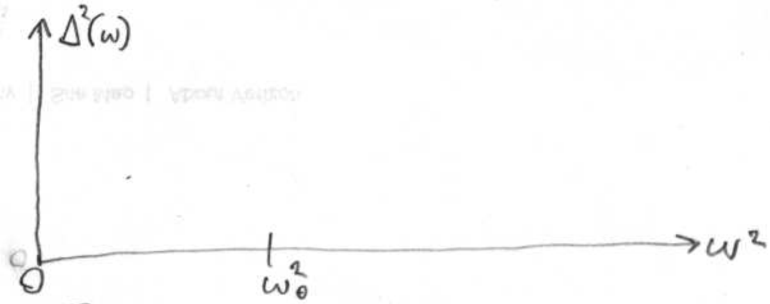
Find Δ for 3 cases of ω:

- limit $\omega \to 0$ $\Delta =$
- on resonance $\omega = \omega_0$ $\Delta =$
- high freq $\omega \gg \omega_0$ $\Delta \approx$

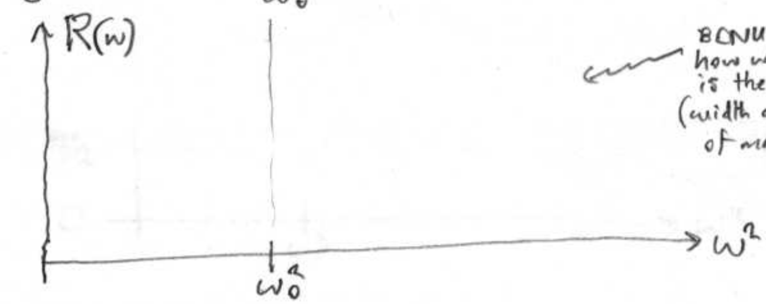
what is ratio of ampl. R between first 2 cases? ratio =

How does this relate to Q?

Sketch Δ^2 vs ω^2:

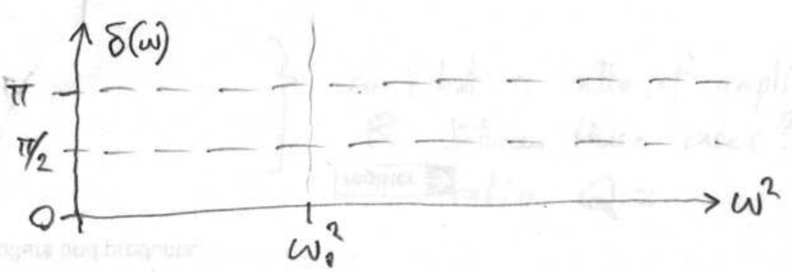


From this sketch R vs ω^2:



BONUS: how wide is the peak? (width at $\frac{1}{\sqrt{2}}$ of max).

From $\tan\delta$ given, sketch δ:



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~ SOLUTIONS ~ (see Q factor sheet too).

FACTS

Driven system $mu'' + \gamma u' + ku = F\cos\omega t$ — F: driving ampl., ω: driving freq. — natural freq $\omega_0 = \sqrt{\frac{k}{m}}$

has steady-state response $U(t) = R\cos(\omega t - \delta)$

ampl. $R = \frac{F}{\Delta}$, with $\Delta^2(\omega) = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$

phase $\tan\delta = \frac{\gamma}{m}\frac{\omega}{\omega_0^2 - \omega^2}$

Find Δ for 3 cases of ω:

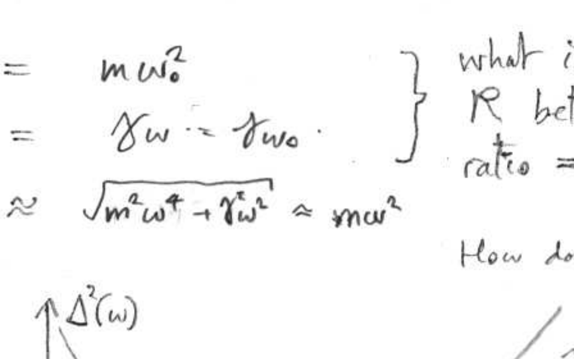
- limit $\omega \to 0$ — $\Delta = m\omega_0^2$
- on resonance $\omega = \omega_0$ — $\Delta = \gamma\omega = \gamma\omega_0$
- high freq $\omega \gg \omega_0$ — $\Delta \approx \sqrt{m^2\omega^4 + \gamma^2\omega^2} \approx m\omega^2$

what is ratio of ampl. R between first 2 cases? ratio $= \frac{m\omega_0^2}{\gamma\omega_0} = Q$

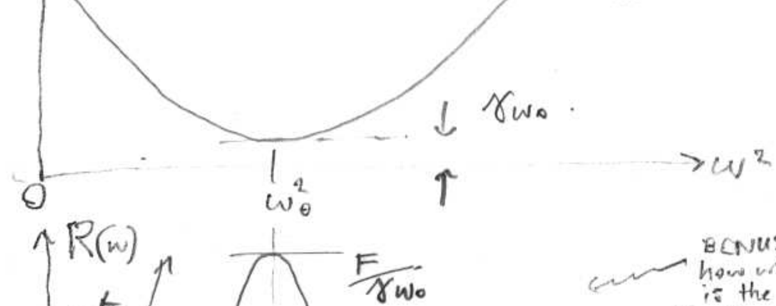
How does this relate to Q? equal to it.

Sketch Δ^2 vs ω^2:

METHOD: Pretend γ v. small, then $m^2(\omega_0^2 - \omega^2)^2$ defines a parabola vs ω^2, with bottom at ω_0^2. Then add a little $\gamma^2\omega^2$.



From this sketch R vs ω^2:



Note: peak is not precisely at $\omega = \omega_0$ since peak is where $\frac{d}{d(\omega^2)}\Delta^2 = -2m^2(\omega_0^2 - \omega^2) + \gamma^2 = 0$

ie $\omega_{max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{1}{2Q}\right)$ — small

BONUS: how wide is the peak? (width at $\frac{1}{\sqrt{2}}$ of max).

here $\Delta^2(\omega) = 2\Delta^2(\omega_0)$ ie $m(\omega_0^2 - \omega^2) = \gamma\omega_0$

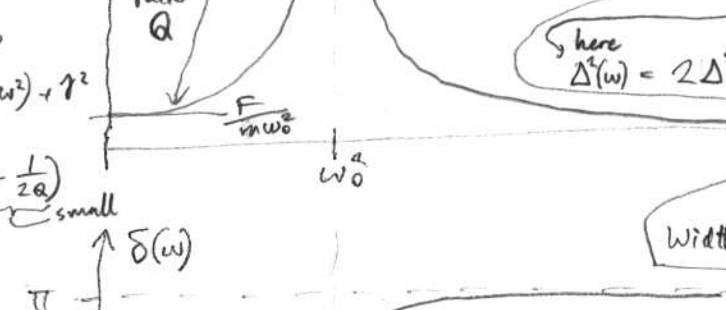
ie $\omega^2 = \omega_0^2(1 \pm$

Width is $2\omega_0^2/Q$.

From $\tan\delta$ given, sketch δ:

$\omega \approx 0$: $\tan\delta \approx 0$ but >0.

$\omega \to \omega_0$: $\tan\delta \to +\infty$ ie $\delta \to \pi/2$.



$\omega \gg \omega_0$: $\tan\delta$ tends to zero from below, but we're already passed through $\pi/2$, so we're approaching π (see 3.8.7)