Examples of Gaussian Elimination

Example 1: Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5. \end{aligned}$$

Solution: We carry out the elimination procedure on both the system of equations and the corresponding augmented matrix, simultaneously. In general only one set of reductions is necessary, and the latter (dealing with matrices only) is preferable because of the simplified notation.

$$\begin{array}{rcrcr} x_1 & + & 5x_2 & = & 7 \\ -2x_1 & - & 7x_2 & = & -5 \end{array} \qquad \begin{pmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{pmatrix}$$

Add twice Row 1 to Row 2.

$$\begin{array}{rcrcr} x_1 & + & 5x_2 & = & 7 \\ & & 3x_2 & = & 9 \end{array} \qquad \begin{pmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{pmatrix}$$

Multiply Row 2 by 1/3.

$$\begin{array}{rcrcr} x_1 & + & 5x_2 & = & 7 \\ & & x_2 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

Add -5 times Row 2 to Row 1.

$$\begin{array}{rcrcr} x_1 & & & = & -8 \\ & & x_2 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{pmatrix}$$

Example 2: Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

Solution: As before, we carry out reduction on the system of equations and on the augmented matrix simultaneously, in order to make it clear that row operations on equations correspond exactly to row operations on matrices.

$$\begin{array}{rcrcrcr} & & 2x_2 & + & x_3 & = & -8 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Swap Row 1 and Row 2.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Add Row 1 to Row 3.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ & - & x_2 & - & x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

Swap Row 2 and Row 3.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & - & x_2 & - & x_3 & = & 3 \\ & & 2x_2 & + & x_3 & = & -8 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{pmatrix}$$

Add twice Row 2 to Row 3.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & - & x_2 & - & x_3 & = & 3 \\ & & & - & x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

Add -1 times Row 3 to Row 2.
Add -3 times Row 3 to Row 1.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & 6 \\ & - & x_2 & & & = & 5 \\ & & & - & x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & -2 & 0 & 6 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

Add -2 times Row 2 to Row 1.

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -4 \\ & - & x_2 & & & = & 5 \\ & & & - & x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

Multiply Rows 2 and 3 by -1.

$$\begin{array}{ccccc} x_1 & & & = & -4 \\ & x_2 & & = & -5 \\ & & x_3 & = & 2 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

1cm

Example 3: Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= 12 \\ 2x_1 + 4x_2 + 12x_3 &= -17 \\ x_1 - 4x_2 - 12x_3 &= 22. \end{aligned}$$

Solution: In this case, we convert the system to its corresponding augmented matrix, perform the necessary row operations on the matrix alone, and then convert back to equations at the end to identify the solution.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 6x_3 & = & 12 \\ 2x_1 & + & 4x_2 & + & 12x_3 & = & -17 \\ x_1 & - & 4x_2 & - & 12x_3 & = & 22 \end{array} \qquad \begin{pmatrix} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -6 & 12 \\ 0 & 8 & 24 & -41 \\ 0 & -2 & -6 & 10 \end{pmatrix}$$

Add -2 times Row 1 to Row 2.
Add -1 times Row 1 to Row 3.

$$\begin{pmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 8 & 24 & -41 \end{pmatrix}$$

Swap Row 2 and Row 3.

$$\begin{pmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Add 4 times Row 2 to Row 3.

$$\begin{pmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 6x_3 & = & 12 \\ & - & 2x_2 & - & 6x_3 & = & 10 \\ & & & & 0 & = & -1 \end{array}$$

Since the final equation

$$0 = -1$$

cannot be satisfied, this system has no solutions.