Worksheet #10

(1) Find the general solution to the differential equation.

$$y'' + y = \tan t, \qquad 0 < t < \pi/2$$

1- Find homogeneous solution.

look at characteristic eqn.

$r^2 + 1 = 0 \quad \Rightarrow r = \pm i$

$\Rightarrow$ homogeneous solution is $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$
where $y_1(t) = \cos t \quad y_2(t) = \sin t$

2- $\tan t = g(t)$ is not amenable to undetermined coefficients.

$\Rightarrow$ must use variation of parameter

$Y(t) = U_1(t) y_1(t) + U_2(t) y_2(t)$
where $U_1(t), U_2(t)$ are the solutions of

$$U_1'(t) = \frac{-y_2(t) g(t)}{W(y_1, y_2)} \qquad U_2'(t) = \frac{y_1(t) g(t)}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow u_1'(t) = -\frac{\sin^2 t}{\cos t} = -\frac{(1-\cos^2 t)}{\cos t}$$

$$= -\sec t + \cos t$$

$$\Rightarrow u_1(t) = \int(-\sec t + \cos t)\,dt = -\ln|\sec t + \tan t| + \sin t$$

$$u_2'(t) = \frac{\cos t \sin t}{\cos t} = \sin t$$

$$\Rightarrow u_2(t) = \int \sin t\,dt = -\cos t$$

The particular soln is

$$\Rightarrow y(t) = [-\ln|\sec t + \tan t| + \sin t]\cos t - \cos t \sin t$$

$$= -\cos t \ln|\sec t + \tan t|$$

$\Rightarrow$ The general solution is

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|$$

(2) Find the general solution to the differential equation
$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \qquad t > 0$$
where the homogeneous solutions are $y_1(t) = t, \quad y_2(t) = te^t$.

First we need to rewrite equation.
to be in the form $y'' + p(t)y' + q(t)y = g(t)$

Divide by t^2

$$\to \quad y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t^2}\right)y = 2t$$

We are given the homogeneous solns $y_1(t) = t$
$y_2(t) = te^t$.

To find particular solution, we use variation of parameter

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$W(y_1,y_2) = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t + te^t - te^t$$
$$= t^2e^t.$$

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)} = -\frac{te^t(2t)}{t^2e^t} = -2$$

$$\to u_1'(t) = -2 \to u_1(t) = -2t$$

$$u_2'(t) = \frac{y_1(t)\,g(t)}{W(y_1, y_2)} = \frac{2t^2}{t^2 e^t} = 2e^{-t}$$

$$\rightarrow u_2(t) = -2e^{-t}$$

$$\rightarrow y(t) = \underbrace{-2t(t)}_{= u_1(t) y_1(t)} + \underbrace{-2e^{-t}\, te^t}_{u_2(t) y_2(t)}$$

$$= -2t^2 - 2t$$

$\rightarrow$ The general solution is

$$y(t) = c_1 t + c_2 te^t + (-2t^2 - 2t)$$