Worksheet #1~~3~~4

Find the first 4 nonzero terms in the series solution of Airy's equation

$$y'' - xy = 0$$

about x_0.

(1) Let $x_0 = 0$.

Goal: find the 1st 4 non-zero coeffients of the Taylor series soln of The Airy eqn centered at zero.

i.e. find a_n st $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution to the DE.

We must plug this into the DE.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\to y'' - xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

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(2) Let $x_0 = 1$.

Again we seek a soln with a Taylor expansion whose coefficients are to be determined.

ie. Guess $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ now plug into DE & solve for a_n.

$$y'(x) = \sum_{n=1}^{\infty} a_n n (x-1)^{n-1}; \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

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Multiplying in the x. we get

$$y'' - xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

In order to simplify we must first get both series to have x^m terms.

let $m = n-2$
$n = m+2$

let $m = n+1$
$n = m-1$

now DE reads

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0.$$

Remove $m=0$ term so we can write as 1 series.

So the new expression is

$$2(1) a_2 + \sum_{m=1}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

Simplifying we see

$$2a_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1) a_{m+2} - a_{m-1}\right] x^m = 0.$$

This implies $a_2 = 0$.

$\&$ $(m+2)(m+1) a_{m+2} - a_{m-1} = 0$

$$\Rightarrow a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}$$

lets write out some terms

m	a_{m+2}
1	$a_3 = \frac{a_0}{3(2)}$
2	$a_4 = \frac{a_1}{4(3)}$
3	$a_5 = \frac{a_2}{5(4)} = 0$ $\rightarrow$ infact $a_8 = a_{11} = \cdots = 0$
4	$a_6 = \frac{a_3}{6(5)} = \frac{a_0}{6(5)(3)(2)}$
5	$a_7 = \frac{a_4}{7(6)} = \frac{a_1}{7(6)(4)(3)}$
7	$a_9 = \frac{a_6}{9(8)} = \frac{a_0}{9(8)(6)(5)(3)(2)}$
8	$a_{10} = \frac{a_7}{10(9)} = \frac{a_0}{10(9)(7)(6)(4)(3)}$

$$\rightarrow y(x) = a_0\left(1 + \frac{x^3}{3(2)} + \frac{x^6}{6(5)(3)(2)} + \frac{x^9}{9(8)(6)(5)(3)(2)} + \cdots\right)$$

$$+ a_1\left(x + \frac{x^2}{4(3)} + \frac{x^7}{7(6)(4)(3)} + \frac{x^{10}}{10(9)(7)(6)(4)(3)} + \cdots\right)$$

Plugging the series into the DE. we get

$$y'' - x\,y = \sum_{n=2}^{\infty} n(n-1)(x-1)^{n-2} a_n - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Note we cannot just multiply x into the series.
we must add zero.
ie $x = (x-1)+1$

→ The expression can be written

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Now we can multiply $(x-1)$ into series.

$$\sum_{n=2}^{\infty} n(n-1)(x-1)^{n-2} - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

This is better but we still cannot write as 1 series. 1st we must make all the $(x-1)$ terms have the same exponent.

let $m = n-2$
$n = m+2$

let $m = n+1$
$n = m-1$

let $m = n$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} (x-1)^m - \sum_{m=1}^{\infty} a_{m-1} (x-1)^m - \sum_{m=0}^{\infty} a_m (x-1)^m = 0.$$

Indexing does not start at the same place.
So we must take out all $m=0$ terms.

So the series expression for the DE is now.

$$2(1)a_2 - a_0 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m - \sum_{m=1}^{\infty} a_{m-1}(x-1)^m$$

$$-\sum_{m=1}^{\infty} a_m (x-1)^m = 0.$$

Now we can expres everything w/ 1 series.

$$2a_2 - a_0 + \sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2} - a_{m-1} - a_m\right](x-1)^m = 0$$

$\rightarrow$ $2a_2 - a_0 = 0 \rightarrow a_2 = \frac{1}{2}a_0$

& $\left[(m+2)(m+1)a_{m+2} - a_{m-1} - a_m\right] = 0$ is the recurrence Relation.

$$\rightarrow \quad a_{m+2} = \frac{a_m + a_{m-1}}{(m+2)(m+1)}$$

Now lets make a table to get terms.

m	a_{m+2}
1	$a_3 = \frac{a_1 + a_0}{3(2)}$
2	$a_4 = \frac{a_2 + a_1}{4(3)} = \frac{a_2}{4(3)} + \frac{a_1}{4(3)} = \frac{a_0}{4(3)(2)} + \frac{a_1}{4(3)}$
3	$a_5 = \frac{a_4 + a_3}{5(4)} = \ldots$
$\vdots$	$\vdots$

[We only need 1st 4 non zero terms.]

now $y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \ldots$

$$= a_0 + a_1(x-1) + \frac{1}{2}a_0(x-1)^2 + \left(\frac{a_1}{6} + \frac{a_0}{6}\right)(x-1)^3 + \left(\frac{a_0}{24} + \frac{a_1}{12}\right)(x-1)^4 + \ldots$$

Grouping by a_0 & a_1 to create the two fundamental Solutions. We get

$$y(x) = a_0\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \ldots\right) + a_1\left((x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \ldots\right)$$