Worksheet #19

(1) Find all the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$$

1st Find eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - 8 = \lambda^2 - 3\lambda + \lambda - 3 - 8$$

$$= \lambda^2 - 2\lambda - 11 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-4(-11)}}{2} = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm 4\sqrt{3}}{2} = 1 \pm 2\sqrt{3}$$

eigenvalues are $\lambda_{1,2} = 1 \pm 2\sqrt{3}$

1st find eigenvector for $\lambda_1 = 1 + 2\sqrt{3}$

$$\left[\begin{array}{cc|c} 3-(1+2\sqrt{3}) & 2 & 0 \\ 4 & -1-(1+2\sqrt{3}) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2-2\sqrt{3} & 2 & 0 \\ 4 & -2-2\sqrt{3} & 0 \end{array}\right]$$

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(2) Verify that $x^1(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$ and $x^2(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$ are solutions to the first order linear equation

$$x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x.$$

Let c_1 and c_2 are arbitrary constants. Verify that $x(t) = c_1x^1 + c_2x^2$ is a solution to the first order system.

$$\bar{x}^{1\prime}(t) = \begin{bmatrix} 3e^{3t} \\ 6e^{3t} \end{bmatrix} \quad \text{Now} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} e^{3t} + 2e^{3t} \\ 4e^{3t} + 2e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 6e^{3t} \end{bmatrix} \checkmark$$

$$\bar{x}^{2\prime}(t) = \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix} \quad \text{Now} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} - 2e^{-t} \\ 4e^{-t} - 2e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix} \checkmark$$

$$\text{Now} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}(c_1\bar{x}^1 + c_2\bar{x}^2) = c_1\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{x}^1 + c_2\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{x}^2$$

$$= c_1\bar{x}^{1\prime} + c_2\bar{x}^{2\prime} = \bar{x}' \checkmark$$

$$① = \frac{2+2\sqrt{3}}{\underbrace{4-4(3)}_{-8}}① \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{4}(2+2\sqrt{3}) & 0 \\ 4 & -2-2\sqrt{3} & 0 \end{array}\right]$$

$$② = ② - 4① \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{4}(2+2\sqrt{3}) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad x_1 = \frac{1}{4}(2+2\sqrt{3})x_2$$

let $x_2 = 4 \Rightarrow$ The eigenvector $\bar{X}_1 = \begin{pmatrix} 2+2\sqrt{3} \\ 4 \end{pmatrix}$

For $\lambda_2 = 1-2\sqrt{3}$, we do row reduction on

$$\left[\begin{array}{cc|c} 3-(1-2\sqrt{3}) & 2 & 0 \\ 4 & -1-(1-2\sqrt{3}) & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2+2\sqrt{3} & 2 & 0 \\ 4 & -2+2\sqrt{3} & 0 \end{array}\right]$$

$$① = \frac{2-2\sqrt{3}}{-8}① \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{4}(2-2\sqrt{3}) & 0 \\ 4 & -2+2\sqrt{3} & 0 \end{array}\right]$$

$$② = ② - 4① \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{4}(2-2\sqrt{3}) & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = \frac{1}{4}(2-2\sqrt{3})x_2 \\ \text{let } x_2 = 4 \end{array}$$

$$\rightarrow \bar{X}_2 = \begin{pmatrix} 2-2\sqrt{3} \\ 4 \end{pmatrix}$$

is the eigenvector.