Worksheet #23 Solns

we must look at

1) $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$

Note $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

$\Rightarrow \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(\sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right)\right)$

If $m \neq n$

$\rightarrow \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} \left[\sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right)\right] dx$

$= \frac{1}{2}\left[\frac{L}{(n+m)\pi} \underbrace{\cos\left(\frac{(m+n)\pi x}{L}\right)\Big|_{-L}^{L}}_{=0} + \frac{L}{(m-n)\pi} \underbrace{\cos\left(\frac{(m-n)\pi x}{L}\right)\Big|_{-L}^{L}}_{=0}\right]$

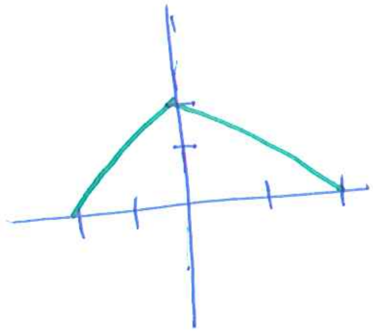
$= 0$

If $m = n$

$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} \int_{-L}^{L} \left[\sin\left(\frac{2m\pi x}{L}\right) + 0\right] dx$

$= \frac{1}{2}\left(\frac{L}{2m\pi} \cos\left(\frac{2m\pi x}{L}\right)\Big|_{-L}^{L}\right) = 0$

2)



$- = f(x)$

$f(x)$ has period 4

& is even.

$\Rightarrow f(x)$ has a cosine series.

$$\rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$L = \frac{T}{2} = 2$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{2}\right)dx \qquad \begin{array}{ll} u = 2-x & v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) \\ du = -dx & dv = \cos\left(\frac{n\pi x}{L}\right)dx \end{array}$$

$$= \frac{2}{\cancel{2}}\int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= (2-x)\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2\right) - \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)(-1)\,dx$$

$$= (2-x)\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2\right) + \left(\frac{2}{n\pi}\right)^2\left(-\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_0^2$$

$$= \cancel{(2-2)(0)} - 2\cancel{\left(\frac{2}{n\pi}\,0\right)}_{0} + \left(\frac{2}{n\pi}\right)^2(-1)\Big(\underbrace{\cos(n\pi)}_{(-1)^n} - 1\Big)$$

$$= \left(\frac{2}{n\pi}\right)^2(-1)\underbrace{\left((-1)^n - 1\right)}_{= \begin{cases} 0 \text{ if } n \text{ even} \\ -2 \text{ if } n \text{ odd} \end{cases}} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{8}{n\pi} & \text{if } n \text{ odd.} \end{cases}$$

$\Rightarrow$ ~~f(x) =~~

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \frac{2}{2}\int_0^2 (2-x)\,dx$$

$$= 2x - \frac{x^2}{2}\Big|_0^2 = 4 - \frac{4}{2} - 0 = 2.$$

$$\Rightarrow f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{8}{(2n-1)\pi} \cos\left(\frac{n\pi x}{2}\right)$$

Alternatively.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)dx$$

$b_n = 0 \quad \forall n.$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)dx$ ☆

$a_0 = 2.$ as before.

ie. Both ways give you the same series.