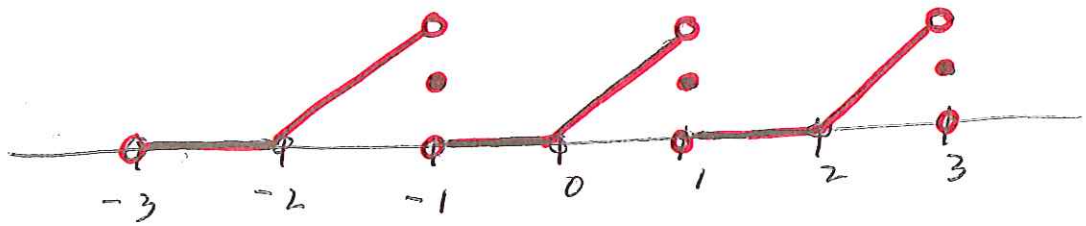
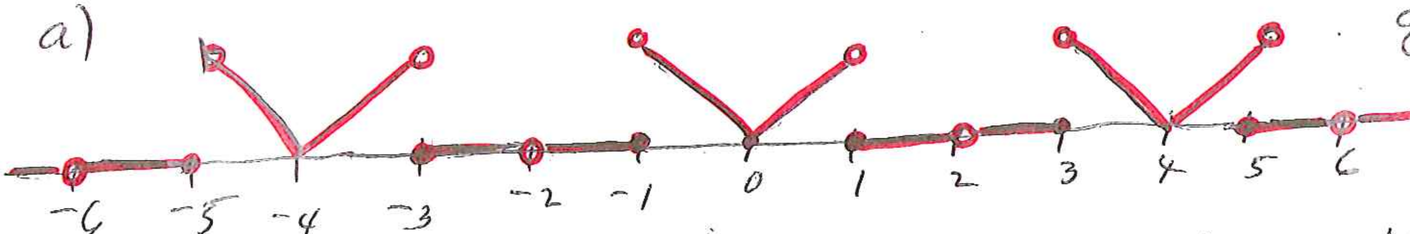
Solutions to Worksheet 24

① $f(x) = \begin{cases} 0 & -1 \le x < 0 \\ x & 0 \le x < 1 \end{cases}$



② $f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$

a)



b) The Fourier cosine series for f is the Fourier series for g (~~viewed on~~ restricted to the interval $[0,2]$.)

g has period 4 ($L=2$).

So ~~the~~ $a_n = \frac{1}{2}\int_{-2}^{2} g(x)\cos\left(\frac{n\pi x}{2}\right)dx$

$= \frac{2}{2}\int_0^2 g(x)\cos\left(\frac{n\pi x}{2}\right)dx$

since g is even

$= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$

since $g = f$ on $[0,2]$.

$\frac{2}{L}$

(Note: Normally you will go directly to writing $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

I did the details in this example just to illustrate why this is the appropriate expression.)

We have $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx$

$= \int_0^1 x\,dx$ (since $f \equiv 0$ on $[1,2)$

$= \frac{1}{2}$

$a_n = \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx$

Integ. by parts: $u = x$ $dv = \cos\left(\frac{n\pi x}{2}\right)$

$du = dx$ $v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

$a_n = \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \int_0^1 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$

$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_0^1 = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$

For
$$\begin{cases} n = 1, 5, 9, \dots & a_n = \frac{2}{n\pi} \\ n = 2, 6, 10, \dots & a_n = \frac{4}{n^2\pi^2}(-1-1) = \frac{-8}{n^2\pi^2} \\ n = 3, 7, 11, \dots & a_n = -\frac{2}{n\pi} \\ n = 4, 8, 12, \dots & a_n = 0 \end{cases}$$

Answer can be expressed as:

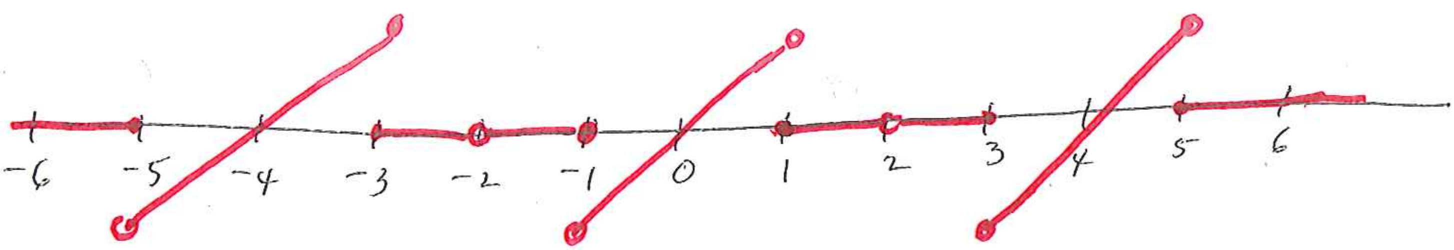
$$\frac{1}{4} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \right) \cos\left(\frac{n\pi x}{2}\right)$$

or as

$$\frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\pi} \cos\left(\frac{(2n-1)\pi x}{2}\right) +$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^2\pi^2} \cos\left(\frac{2n\pi x}{2}\right)$$

c) $h(x) =$ odd extension



d) $b_n = \frac{2}{L} \int_0^L \cancel{\sin}\, f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$u = x \quad dv = \sin\left(\frac{n\pi x}{2}\right) \quad du = dx \quad \cancel{dv}\, v = -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)$

$$b_n = \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \int_0^1 \frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{-2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$

$$= \frac{-2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

When $n=2m$ even, $\cos\left(\frac{n\pi}{2}\right) = \cos(m\pi) = (-1)^m$

$\sin\left(\frac{n\pi}{2}\right) = \sin(m\pi) = 0$

When $n = 2m-1$ odd.

$\cos\left(\frac{n\pi}{2}\right) = 0 \qquad \sin\left(\frac{n\pi}{2}\right) = (-1)^{m+1}$

So $b_{2m} = \frac{(-2)(-1)^m}{2m\pi} = \frac{(-1)^{m+1}}{m\pi}$

$b_{2m-1} = \frac{4}{(2m)^2\pi^2} (-1)^{m+1} = \frac{(-1)^{m+1}}{m^2\pi^2}$

Series

$$\sum_{m=1}^{\infty} \left(\frac{(-1)^{m+1}}{m^2\pi^2} \sin\left(\frac{(2m-1)\pi x}{2}\right) + \right.$$

(*)

$$\left. \frac{(-1)^{m+1}}{m^2\pi^2} \sin\left(\frac{2m\pi x}{2}\right) \right)$$

(You can also write

$$\sum_{n=1}^{\infty} \left[\frac{-2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right] \sin\left(\frac{n\pi x}{2}\right)$$

although (*) makes it clearer.)