Worksheet #25

(1) Are the following PDEs separable? If so, find the two differential equations to replace the PDE.

- $tu_{xx} + xu_t = 0$

let $U(x,t) = X(x)T(t)$ separate

Plug in $tX''T + xXT' = 0 \Rightarrow \frac{X''}{xX} = \frac{T'}{tT} = -\lambda$ $\lambda > 0$ constant.

$\Rightarrow X'' + x\lambda X = 0 \quad T' + \lambda t T = 0 \Rightarrow$ Separable.

- $(x + y^2)u_{xx} + u_{yy} = 0$

let $U(x,y) = X(x) \cdot Y(y)$

Plugin $\Rightarrow (x+y^2)X''Y + XY'' = 0$ Try to separable.

$xX''Y + y^2 X''Y + XY'' = 0$

Cannot separate $\Rightarrow$ Not separable.

(2) Find the solution to the following heat conduction problem

$$100u_{xx} = u_t \qquad 0 < x < 1, \quad t > 0$$
$$(*) \quad u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$
$$u(x,0) = \sin(2\pi x) - \sin(5\pi x), \quad 0 \le x \le 1$$

let $U(x,t) = X(x)T(t)$.

Pluginto PDE. $100X''(x)T(t) = T'(t)X(x)$

separate $\frac{X''}{X} = \frac{T'}{100T} = -\lambda$

$\Rightarrow$ $X'' + \lambda X = 0$

$T' + \lambda 100 T = 0 \Rightarrow T = Ce^{-\lambda 100 t}$

$\Rightarrow X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

By $(*)$ $u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \Rightarrow C_1 = 0$.

$U(1,t) = X(1)T(t) = 0 \Rightarrow \sin(\sqrt{\lambda}) = 0$

$\Rightarrow \sqrt{\lambda} = n\pi$ $n = 1, 2, \ldots$

$\Rightarrow \lambda = (n\pi)^2$

$\Rightarrow X_n(x) = \sin(n\pi x)$

$\Rightarrow U_n(x,t) = C_n e^{(n\pi)^2 100t} \sin(n\pi x)$

$\Rightarrow U(x,t) = \sum_{n=1}^{\infty} C_n e^{(n\pi)^2 100t} \sin(n\pi x)$

$$U(x,0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) = \sin(2\pi x) - \sin(5\pi x)$$

$$C_n = \frac{1}{1}\int_{0}^{L} f(x)\sin(n\pi x)\, dx$$

$$= \int_0^1 [\sin(2\pi x) - \sin(5\pi x)]\sin(n\pi x)\, dx$$

$$= \left[\int_0^1 \sin(2\pi x)\sin(n\pi x)\, dx + (-1)\int_0^1 \sin(5\pi x)\sin(n\pi x)\, dx\right]$$

$$= \begin{cases} 1 & \text{for } n = 2 \\ -1 & \text{for } n = 5 \\ 0 & \text{otherwise} \end{cases}$$

by orthogonality of sine functions.

$$\Rightarrow U(x,t) = e^{-4\pi^2(100)t}\sin(2\pi x) - e^{-25\pi^2(100)t}\sin(5\pi x)$$