Worksheet #26

(1) Find the steady-state solution to the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies

$$u_x(0,t) - u(0,t) = 0 \qquad u(L,t) = \cancel{0}T.$$

let $u(x,t) = V(x) + W(x,t)$

Plug into PDE $\Rightarrow$ $V''(x) = 0$ $\quad W_t = \alpha^2 W_{xx}$

we want homogeneous BC for W.

$u_x(0,t) - u(0,t) = V'(0) + W_x(0,t) - V(0) - W(0,t) = 0$

let $W_x(0,t) = W(0,t) = 0$.

$\rightarrow V'(0) = V(0)$

$U(L,t) = V(L) + W(L,t) = 0 \xrightarrow{\text{force}} W(L,t) = 0 \quad \rightarrow V(L) = L$

Now $V(x) = C_1 x + C_2 \qquad V'(0) = C_1 = C_2 = V(0)$

(2) Consider a rod of length 30 for which $\alpha^2 = 1$. Suppose the initial temperature distribution is given by $u(x,0) = \dfrac{x(60-x)}{30}$ and the boundary conditions are $u(0,t) = 30$ and $u(30,t) = 0$. Find the temperature as a function of position and time.

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 30 \\ u(30,t) = 0 \\ u(x,0) = \dfrac{x(60-x)}{30} \end{cases}$$

let $u(x,t) = V(x) + W(x,t)$

Goal: Choose $V(x)$ st $W(x,t)$ satisfies a homogeneous BC problem

Plug into PDE. & choose

$V''(x) = 0 \qquad W_t = W_{xx}$

Now look at BC.

$U(0,t) = 30 = V(0) + W(0,t)$

we want $W(0,t) = 0 \rightarrow V(0) = 30$

$U(30,t) = 0 = V(30) + W(30,t)$

we want $W(30,t) = 0 \rightarrow V(30) = 0.$

#1 continued

$$V(L) = c_1 L + c_2 = T$$

$$\Rightarrow c_1(L+1) = T$$

$$\Rightarrow c_1 = \frac{T}{L+1}$$

$$\Rightarrow V(x) = \frac{T}{L+1}(x+1)$$

The steady state solution

$$U(x) = \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} V(x) + \underbrace{W(x,t)}_{\to 0} = V(x)$$

$$= \frac{T}{L+1}(x+1)$$

So far we have

$v''(x) = 0$

$v(0) = 30$

$v(30) = 0$

$w_t = w_{xx}$

$w(0,t) = w(30,t) = 0$

We need IC for $w(x,t)$

$$u(x,0) = \frac{x(160-x)}{30} = v(x) + w(x,0)$$

$$\Rightarrow w(x,0) = \frac{x(160-x)}{30} - v(x).$$

1st Find $v(x)$.

$v(x) = c_1 x + c_2$

$v(0) = 30 = c_2$

$v(30) = 30c_1 + 30 = 0 \Rightarrow c_1 = -1$

$\Rightarrow v(x) = 30 - x$ It is the solution of.

Now for $w(x,t)$.

$$\begin{cases} w_t = w_{xx} \\ w(0,t) = w(30,t) = 0 \\ w(x,0) = \dfrac{x(160-x)}{30} + x - 30 \end{cases}$$

Because we have homogeneous BC for the temp. W.

we know it $w(x,t)$ can be expressed as a sin series.

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{30}\right)^2 t} \sin\left(\frac{n\pi}{30}x\right)$$

$$\Rightarrow c_n = \frac{2}{30}\int_0^{30}\left[\frac{x(160-x)}{30} + x - 30\right]\sin\left(\frac{n\pi x}{30}\right)dx$$

C_n Integration by parts

$$u = \left[\frac{x(60-x)}{30} + x - 30\right]$$

$$du = \left(\frac{60-x}{30} + 1\right) = -\frac{x}{30} + 3$$

$$v = -\frac{30}{n\pi}\cos\left(\frac{n\pi x}{30}\right)$$

$$dv = \sin\left(\frac{n\pi x}{30}\right)dx$$

$$C_n = \frac{1}{15}\left[\overset{u}{\left(\frac{x(60-x)}{30} + x - 30\right)}\overset{v}{\left(\frac{-30}{n\pi}\right)\cos\left(\frac{n\pi x}{30}\right)}\Bigg|_0^{30} + \frac{30}{n\pi}\int_0^{30}\left(3 - \frac{x}{30}\right)\cos\left(\frac{n\pi x}{30}\right)dx\right]$$

$$-\int v\,du$$

$$= \frac{1}{15}\left[\frac{-(30)^2}{n\pi}\cos(n\pi) + \frac{(30)^2}{n\pi} + \frac{30}{n\pi}\int_0^{30}\left(3 - \frac{x}{30}\right)\cos\left(\frac{n\pi x}{30}\right)dx\right]$$

$$u = 3 - \frac{x}{30} \qquad v = \frac{30}{n\pi}\sin\left(\frac{n\pi x}{30}\right)$$

$$du = -\frac{1}{30}dx \qquad dv = \cos\left(\frac{n\pi x}{30}\right)dx$$

$$= \frac{1}{15}\left[\frac{-(30)^2}{n\pi}\cos(n\pi) + \frac{(30)^2}{n\pi} + \frac{30}{n\pi}\left[\overset{uv}{\left(3 - \frac{x}{30}\right)\left(\frac{30}{n\pi}\right)\sin\left(\frac{n\pi x}{30}\right)}\Bigg|_0^{30} - \frac{30}{n\pi}\int_0^{30} -\frac{1}{30}\sin\left(\frac{n\pi x}{30}\right)dx\right]\right]$$

$$\searrow 0$$

$$= \frac{1}{15}\left[\frac{-(30)^2}{n\pi}\cos(n\pi) + \frac{(30)^2}{n\pi} - \frac{30}{(n\pi)^3}\cos\left(\frac{n\pi x}{30}\right)\Bigg|_0^{30}\right]$$

$$= \frac{1}{15}\left[\frac{-(30)^2}{n\pi}\cos(n\pi) + \frac{(30)^2}{n\pi} - \frac{30}{(n\pi)^3}\left[\cos(n\pi) - 1\right]\right]$$