Worksheet #2

Part 1

First order differential equations in which the independent variable does not appear explicitly, ie.

$$\frac{dy}{dt} = f(y),$$

are called *autonomous*. Equations of this form can be used to model population growth or decline.

Let $y(t)$ denote a population of a given species at a time t.

(1) Suppose a population grows at a rate that is proportional to the the current population, ie.

$$\frac{dy}{dt} = ry.$$

(a) Subject to an initial condition $y(0) = y_0$, find the solution to the initial value problem.

Separate $\frac{dy}{y} = r\,dt$

Integrate $\ln y = rt + C$

$\Rightarrow y = Ce^{rt}$ $\quad y(0) = C = y_0 \Rightarrow y(t) = y_0 e^{rt}$

(b) Why are populations that can be modeled by the differential equation said to have exponential growth?

The population has exponental growth because the solution to the DE grows exponentially with respect to t for r positive.

(2) Suppose a population can be modeled by the *logistic equation*,

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y.$$

(a) Solve the logistic equation when $r = 3$ and $K = 1$.

$\frac{dy}{dt} = 3(1-y)y$

Separate $\frac{dy}{y(1-y)} = 3\,dt$

$\rightarrow \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = 3t$

Integrate $\ln(y) - \ln(1-y) = 3t + C$

Partial fractions.

$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$

$= \frac{A - Ay + By}{y(1-y)}$

$\rightarrow A = 1 \quad 0 = B - A$

$\rightarrow B = 1$

(b) What are the critical points (or equillibrium)?

critical points are where $\frac{dy}{dt} = 0$.

ie. $y(1-y) = 0 \rightarrow$ critical pts are $y = 0$, & $y = 1$

Part 2

(1) Determine an interval in which the solution of the following initial value problem is certain to exist.

$$t(t-4)y' + y = 0, \qquad y(2) = 1$$

$y' + \frac{1}{t(t-4)}y = 0$. $\rightarrow$ Use Thm 2.4.1

$P(t) = \frac{1}{t(t-4)}$ is continuou for $t \neq 0$ & $t \neq 4$.

Since we want a continuous solution & IC is at $t = 2$.
the interval we are looking for is $I = (0, 4)$

(2) Consider the differential equation

$$(3y - y^2)\frac{dy}{dt} = 1 + t^2.$$

At which points in the t, y plane are not gauranteed the existence and uniqueness of a solution.

$y' = \frac{1+t^2}{(3y - y^2)} = f(t,y)$ Use Thm 2.4.2.

$f(t,y)$ is not continuous when $y = 0$ or 3.
$f_y(t,y)$: will not be continuous at the same pts.

$\rightarrow$ The solution is not gauranteed to be unique for $\forall t$ st $y = 0$ or $y = 3$.