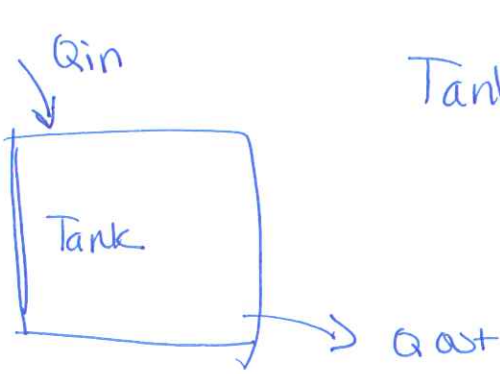
1)



Tank Volume = 100 gal.

let Q(t) be the concentration of salt at time t.

Total rate = rate in − rate out = oz/min

$\underline{\text{Rate in}}$
$Q_{in} = \frac{1}{2}\left(1+\frac{1}{2}\sin t\right)\frac{oz}{gal} \cdot 2\,\frac{gal}{min}$

$= 1 + \frac{1}{2}\sin t \;\; oz/min.$

$\underline{\text{Rate out}}$
$Q_{out} = \frac{Q\,oz}{100\,gal}\;\frac{2\,gal}{min} = \frac{Q}{50}\;\frac{oz}{min}$

$$\begin{cases} Q'(t) = 1 + \frac{1}{2}\sin t - \frac{Q}{50} \\ Q(0) = 50\,oz \end{cases}$$

Solve ~~Ba~~ IVP.

$$Q' - \frac{Q}{50} = 1 + \tfrac{1}{2}\sin t$$

use integrating factor.

1-find μ $\quad \mu(t) = e^{\int -1/50\, dt} = e^{-1/50\, t}$

2- multiply.

$$e^{-\frac{1}{50}t} Q' - \frac{1}{50} e^{-\frac{1}{50}t} Q = \left(1 + \tfrac{1}{2}\sin t\right) e^{-1/50\, t}.$$

3- Rewrite

$$\frac{d}{dt}\left(e^{-\frac{1}{50}t} Q\right) = e^{-\frac{1}{50}t} + \frac{1}{2}\sin t\, e^{-\frac{1}{50}t}$$

4 - Integrate

$$e^{-\frac{1}{50}t} Q = -50\, e^{-\frac{1}{50}t} + \frac{1}{2}\underbrace{\int \sin t\, e^{-\frac{1}{50}t}\, dt}_{\text{Integrate by parts.}}$$

$$\int \sin t\, e^{-\frac{1}{50}t}\, dt$$

$u = \sin t \qquad v = -50\, e^{-1/50\, t}$

$du = \cos t\, dt \qquad dv = e^{-\frac{1}{50}t}\, dt$

$$= \sin t\left(-50\, e^{-\frac{1}{50}t}\right) + 50 \int \cos t\, e^{-\frac{1}{50}t}\, dt$$

$v = -50\, e^{-\frac{1}{50}t}$

$u = \cos t$

$du = -\sin t\, dt \qquad dv = e^{-\frac{1}{50}t}\, dt$

$$= \sin t\left(-50\, e^{-\frac{1}{50}t}\right) + 50\left(\cos t\left(-50\, e^{-\frac{1}{50}t}\right) - 50 \int \sin t\, e^{-\frac{1}{50}t}\, dt\right)$$

collect like terms.

$$\left(1 + 50^2\right)\int \sin t\, e^{-\frac{1}{50}t}\, dt = -50 \sin t\, e^{-\frac{1}{50}t} - 50^2 \cos t\, e^{-\frac{1}{50}t}$$

$$\rightarrow e^{-\frac{1}{50}t} Q = \left(-50 - \frac{50}{(1+50)^2}\left(\sin t - 50\cos t\right)\right)e^{-\frac{1}{50}t} + C$$

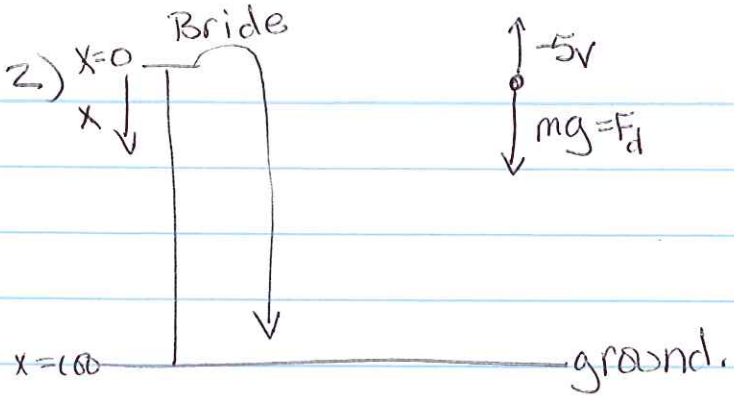
$$\rightarrow Q(t) = -50\left(1 + \frac{1}{(1+50^2)}\left(\sin t - 50\cos t\right)\right) + Ce^{-1/50\, t}$$

Now Find the constant using the intial condition.

$$Q(0) = -50\left(1 + \frac{1}{(1+50^2)}(-50)\right) + C = 50.$$

$$\rightarrow C = \frac{50}{1+50^2}$$

$$\rightarrow Q(t) = -50\left(1 + \frac{1}{1+50^2}\left(\sin t - 50\cos t\right)\right) + \frac{50}{1+50^2}e^{-\frac{1}{50}t}$$

2)


$$\begin{cases} mv' = mg - 5v \Rightarrow v' = 9.8 - v/10 \\ v(0) = -10 \end{cases}$$

we can solve this using the integrating factor

$$v' + \tfrac{1}{10} v = 9.8$$

1- $\mu(t) = e^{\int 1/10\, dt} = e^{1/10\, t}$

2- multiply $e^{1/10 t} v' + \frac{1}{10} e^{1/10 t} v = 9.8 e^{1/10 t}$

3- rewrite $\frac{d}{dt}\left(e^{1/10 t} v\right) = 9.8 e^{1/10 t}$

4- integrate

$$e^{1/10 t} v = 9.8(10) e^{1/10 t} + C$$

$$\Rightarrow v(t) = \left(98 + C e^{-1/10 t}\right)$$

$$v(0) = 98 + C = -10 \Rightarrow C = -108$$

$$v(t) = 98 - 108 e^{-1/10 t}$$

Position is given by $x(t) = \int v(t)\, dt$

we know $x(0) = 0$.

$$x(t) = 98t + 1080 e^{-1/10 t} + C$$

$$x(0) = 1080 + C = 0 \Rightarrow C = -1080.$$

$$x(t) = 98t + 1080 e^{-1/10 t} - 1080$$