~~Ex~~ a matrix A is invertible if its determinant is non zero.

If a matrix is not invertible then it is called <u>singular</u>.

<u>Ex.</u> let $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$ ~~IS~~

Is A singular?

<u>Linear independence</u>

<u>Recall</u>: What does it mean for 2 functions $f_1(x)$ & $f_2(x)$ to be linearly independent?

$\Rightarrow$ ~~For~~ the only ~~for~~ constants c_1 & c_2 st
$c_1 f_1(x) + c_2 f_2(x) = 0 \Rightarrow c_1 = c_2 = 0$.

Let $\{\bar{x}^1, \ldots \bar{x}^n\}$ be a set of n vectors. of ~~length~~ size $n \times 1$

<u>Def</u>: • $\{\bar{x}^1, \ldots \bar{x}^n\}$ are <u>linearly dependent</u> if there exist a set of constants $c_1, \ldots, c_n$ at least 1 non-zero st
$c_1 \bar{x}^1 + c_2 \bar{x}^2 + \ldots + c_n \bar{x}^n = 0$.

• $\{\bar{x}^1, \ldots, \bar{x}^n\}$ are <u>linearly independent</u> if ~~give~~ for constants $c_1, \ldots, c_n$
& $c_1 \bar{x}^1 + c_2 \bar{x}^2 + \ldots + c_n \bar{x}^n = 0$
$\Rightarrow c_1 = c_2 = \ldots = c_n = 0$.

How do we check if vectors are linearly dependent or independent?

We look at the ~~matr~~ linear system

$$\begin{bmatrix} x_1^1 & \cdots & x_1^n \\ x_2^1 & & x_2^n \\ \vdots & & \vdots \\ x_n^1 & & x_n^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \bar{0}$$

$$\bar{X} \quad \bar{c} = \bar{0}$$

If $\bar{c} = 0$ is the only answer then the vectors are linearly independent.

In otherwords, if $\bar{X}$ is invertible or nonsingular then the vectors are linearly independent.

Ie. We need to look at the $\det(\bar{X})$.

Ex Are the vectors $x^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $x^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $x^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$

linearly dependent or independent?

Now let's talk about eigenvalues and eigenvectors.

~~Consider the equation~~ $\cancel{A\bar{x} = \bar{b}}$

let A be a matrix.

Goal: Find values of λ & vectors $\bar{x} \neq \bar{0}$

st $A\bar{x} = \lambda\bar{x}$

We can rewrite this as

$$A\bar{x} - \lambda\bar{x} = 0$$

$\Rightarrow (A - \lambda I)\bar{x} = 0.$

We know $\bar{x} = 0 \Rightarrow A - \lambda I$ must be singular. $\Rightarrow$ The $\det(A - \lambda I) = 0$.

We call the values of λ st $\det(A - \lambda I) = 0$. the eigenvalues of A.

& The vectors $\bar{x}$ st $A\bar{x} = \lambda\bar{x}$ the eigenvectors.

Ex let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

Find the eigenvalues & eigenvectors of A.

1^{st} Find eigenvalues.

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$$

$$\rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2$$

$$= 6 - 5\lambda + \lambda^2 - 2 = \lambda^2 - 5\lambda + 4$$

We want $\det(A - \lambda I) = 0$

$$\rightarrow \lambda^2 - 5\lambda + 4 = 0 \rightarrow \lambda = +4, 1$$

Now lets find eigen vectors. let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Goal is to find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ when $\lambda = 1$.

$\rightarrow$ Solve $A\bar{x} = \bar{x}$. $\rightarrow (A - I)\bar{x} = 0$

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{(2) = -2(1) + (2)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow x_1 = -x_2 =$$

Choose $x_2 = \alpha \Rightarrow \bar{x} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$.

we are free to chouse

take $\alpha = 1$ $\rightarrow$ $\bar{X} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

verify $A\bar{x} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1(2)+1(1) \\ 2(-1)+3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \bar{x}$ ✓

For $\lambda = 4$

we need to find $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ st

$(A - 4I)\bar{x} = 0.$

$\begin{bmatrix} 2-4 & 1 \\ 2 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use row reduction

$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{① = -\frac{1}{2}①} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right]$

$\xrightarrow{② = -2① + ②} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

~~Now we can read of~~

$\rightarrow x_1 + -1/2\, x_2 = 0$

let $x_2 = \alpha \rightarrow \bar{x} = \begin{pmatrix} 1/2\,\alpha \\ \alpha \end{pmatrix}$

let $\alpha = 2$ $\bar{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ verify.