Goals
- Solve systems of linear equations by row reduction.
- Find inverse matrices by row reduction.
- Check whether a collection of vectors is linearly independent.
- Compute and interpret determinants.

Solving systems of linear equations:

Warm-up example:

1) $x_1 + 5x_2 = 17$
2) $2x_1 + 3x_2 = 13$.

We can write this as $\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$.

Compile this information in the shorthand notation of an "augmented" matrix:

$$\left[\begin{array}{cc|c} 1 & 5 & 17 \\ 2 & 3 & 13 \end{array}\right]$$

(Matrix of coefficients of x_1 & x_2 is on the left, constant column on the right of the line.)

To motivate the method of row reduction, we solve the system in the familiar way, noting the effect at each step on the augmented matrix.

1) $x_1 + 5x_2 = 17$

2) $2x_1 + 3x_2 = 13$.

$$\left[\begin{array}{cc|c} 1 & 5 & 17 \\ 2 & 3 & 13 \end{array}\right]$$

↓ add (-2)(equation 1) to equation 2.

↓ row 2 replaced by (row 2) − 2(row 1)

1) $x_1 + 5x_2 = 17$

2) $-7x_2 = -21$

$$\left[\begin{array}{cc|c} 1 & 5 & 17 \\ 0 & -7 & -21 \end{array}\right]$$

↓ Mult eq. 2 by $(-\frac{1}{7})$

↓ row 2 $\longrightarrow$ $(-\frac{1}{7})$ row 2

1) $x_1 + 5x_2 = 17$

2) $x_2 = 3$

$$\left[\begin{array}{cc|c} 1 & 5 & 17 \\ 0 & 1 & 3 \end{array}\right]$$

↓ Subtract 5(eq. 2) from eq. 1.

↓ row 1 $\longrightarrow$ row 1 − 5 row 2

1) $x_1 = 2$

2) $x_2 = 3$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Read off answer: $x_1 = 2$ $x_2 = 3$.

This example illustrates the type of "row operations" allowed.

- ~~Add mult~~ Replace row (i) by row (i) + c row (j) (c a constant)
  (Leave row (j) alone.)

- Multiply a row by a constant.
- Interchange 2 rows.
  (We didn't do interchanges in the warm-up example, but it just corresponds to reordering the equations.)

~~Goal:~~ <u>Reading off solution</u>

1) If you can row reduce so that you have the identity matrix on the left side of the vertical line, then you have a unique solution and can read it off.

e.g. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$ says $x_1 = 2$, $x_2 = 1$, $x_3 = 4$

2). If you reach a point in which you have a row with all zeroes to the left of the vertical line and a non-zero entry on the right, system is inconsistent, i.e. it has no solution.

e.g. $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 5 \end{array}\right]$ says $0x_1 + 0x_2 + 0x_3 = 5$

$0 = 5$

inconsistent.

3). Suppose ~~yo~~ after row reducing, you obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 1 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We have:

$$x_1 + 2x_2 \quad + 5x_4 = 1$$
$$x_3 + 6x_4 = 2$$

So $x_1 = 1 - 2x_2 - 5x_4$

$x_3 = 2 - 6x_4$

This places no restrictions on x_2 and x_4. So the solution depends on two free parameters.

Write $\alpha = x_2$ $\beta = x_4$ (arbitrary),

we get $x_1 = 1 - 2\alpha - 5\beta$

$x_2 = \alpha$

$x_3 = 2 - 6\beta$

$x_4 = \beta$

Each choice of α, β gives a solution.

These examples motivate the following goals in row reducing:

Row reduce to following form:

- First non-zero entry in each row is 1. These 1's are called "pivots".
- ~~In the columns contain~~ The entries above and below the pivot are zeroes. In particular, a column can have only one pivot.
- (For convenience): The rows are ordered so that the pivots move to the right as you go down.

Examples above: 1) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Every column has a pivot.

3) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 1 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

In this example, columns 1 and 3 have pivots. In our solution above, we expressed x_1 and x_3 (determined ~~Observe:~~ variables) in terms of x_2 and x_4 (free variables).

* The columns with pivots determine variables
* Columns without pivots correspond to free variables

<u>Exercise</u>: Consider both the system

$$\begin{aligned} x_1 + 2x_2 + x_3 + 5x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 + 4x_4 &= 11 \\ x_1 + 2x_2 \qquad\quad - x_4 &= 5 \\ 4x_1 + 8x_2 + 2x_3 + 8x_4 &= 22 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 6 \\ 2 & 4 & 1 & 4 & 11 \\ 1 & 2 & 0 & -1 & 5 \\ 4 & 8 & 2 & 8 & 22 \end{array}\right]$$

pivot is circled

and the system

$$\begin{aligned} x_1 + 2x_2 + x_3 + 5x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 4x_4 &= 0 \\ x_1 + 2x_2 \qquad\quad - x_4 &= 0 \\ 4x_1 + 8x_2 + 2x_3 + 8x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 2 & 4 & 1 & 4 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ 4 & 8 & 2 & 8 & 0 \end{array}\right]$$

We have a pivot in 1,1 spot, so clear rest of the 1st column.

row2 → row2 − 2row1
row3 → row3 − row1
row4 → row4 − 4row1

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 6 \\ 0 & 0 & -1 & -6 & -1 \\ 0 & 0 & -1 & -6 & -1 \\ 0 & 0 & -2 & -12 & -2 \end{array}\right].$$

row2 → (row2)(−1)

pivot

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 6 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & -1 & -6 & -1 \\ 0 & 0 & -2 & -12 & -2 \end{array}\right]$$

clear col 3
row1 → row1 − row2
row3 → row3 + row2
row4 → row4 + 2row2

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 - x_4 = 5$$
$$x_3 + 6x_4 = 1$$

$$x_1 = 5 - 2x_2 + x_4$$
$$x_3 = 1 - 6x_4$$

x_2, x_4 free.

Write $\alpha = x_2$ $\beta = x_4$

$x_1 = 5 - 2\alpha + \beta$, $x_2 = \alpha$,
$x_3 = 1 - 6\beta$, $x_4 = \beta$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 0 & 0 & -1 & -6 & 0 \\ 0 & 0 & -1 & -6 & 0 \\ 0 & 0 & -2 & -12 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 5 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & -1 & -6 & 0 \\ 0 & 0 & -2 & -12 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 - x_4 = 0$$
$$x_3 + 6x_4 = 0$$

$$x_1 = -2x_2 + x_4$$
$$x_3 = -6x_4$$

$x_1 = -2\alpha + \beta$
$x_2 = \alpha$
$x_3 = -6\beta$
$x_4 = \beta$

As expected, we see that the sol'n of the inhomogeneous system is a partic. sol'n $(5,1,0,0)$ + gen'l sol'n of homog system

$$(5,1,0,0) + \underbrace{\alpha(-2,1,0,0) + \beta(1,0,-6,1)}_{\text{sol'n of homog. system}}$$

II. Inverse matrices

Example $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Need $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note that this says

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus we have 2 systems of equations:

$$A\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since only the constant columns (& the names of the variables) differ, we can solve both at once by having one column for each system to the right of the line

General ~~prod~~ procedure for inverting $n \times n$ matrix A:

Begin with $\left[\, A \;\middle|\; I_n \,\right]$

$n \times n$ identity

~~Try to~~ Row reduce. If can row reduce to I_n on left, A^{-1} will be given on the right.

$$\left[\, I_n \;\middle|\; A^{-1} \,\right].$$

Since A is a square matrix, if it doesn't row reduce to the identity, you will necessarily end up with a row of zeroes on the left side:

$$\left[\begin{array}{c|c} X & \text{whatever} \\ 0\ 0 \cdots 0 & \end{array}\right]$$

In that case, A isn't invertible