Notes on Heat Equation on a plate

Let \( \Omega \) be a circle of radius \( r = 1 \). We want to solve

\[
(0.1)\quad u_t = a^2 \Delta u(x) = a^2 \left( u_{rr} + \frac{1}{r} u_r \right) \quad 0 \leq r \leq 1
\]

\[
u(0, t) = 0
\]

\[
u(r, 0) = F(r)
\]

We assume that the solution is separable, ie. \( u(r, t) = R(r)T(t) \). Plugging this into (0.1) we find

\[
RT' = a^2 \left( R'' + \frac{1}{r} R' \right) T
\]

We can separate this equation by grouping \( R \)'s and \( T \)'s.

\[
\frac{T'}{a^2 T} = \frac{R'' + \frac{1}{r} R'}{R} = -\lambda^2
\]

where \( \lambda \) is a constant to be determined.

First solving \( \frac{T'}{a^2 T} = -\lambda^2 \), we find \( T(t) = e^{-a^2 \lambda^2 t} \).

Now we must solve

\[
\frac{R'' + \frac{1}{r} R'}{R} = -\lambda^2.
\]

Putting everything on one side and multiplying by \( r^2 \), we get a second order differential equation

\[
(0.2)\quad r^2 R'' + r R' + \lambda^2 r^2 R = 0
\]

Note that this is very similar to the 0\(^{th} \) order Bessel equation. To see the difference, let’s look for a series solution of the form

\[
R(r) = \sum_{n=1}^{\infty} a_n(k) x^{n+k}.
\]

Plugging this into (0.2), we find the indicial equation is \( k^2 = 0 \), \( a_1 = 0 \) and the recurrence relation for the coefficients is

\[
a_n = -\frac{\lambda^2 a_{n-2}}{n^2}.
\]

Since \( a_1 = 0 \), all odd terms must equal zero.

\[
a_{2m} = \frac{(-1)^m (\lambda^2)^m a_0}{(m!)^2 2^{2m}}
\]

So the series solution is

\[
R_1(r) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^m (\lambda r)^{2m}}{(m!)^2 2^{2m}} = J_0(\lambda r).
\]

Likewise, the second homogeneous solution is given by \( Y_0(\lambda r) \). Thus \( R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r) \). Now \( Y_0 \) blows up at the origin so we must set \( c_2 = 0 \).

Thus \( R(r) = c_1 J_0(\lambda r) \). Hence, \( u(r, t) = c_1 J_0(\lambda r) e^{-a^2 \lambda^2 t} \).

We know that \( u(1, t) = 0 = J_0(\lambda) \). This means that \( \lambda \) must be the roots of \( J_0 \). \( J_0 \) has infinitely many roots thus by superposition
\[ u(r, t) = \sum_{l=0}^{\infty} c_l J_0(\lambda_l r) e^{-a^2 \lambda_l^2 t} \]

The initial condition \( u(r, 0) = F(r) \) determines the coefficients \( c_l \).

The coefficients are (without explanation)

\[ c_l = \frac{2}{J_1^2(\lambda_l)} \int_0^1 x J_0(\lambda_l r) F(r) dr. \]