Series Solutions of Second Order Linear ODEs

Craig J. Sutton

craig.j.sutton@dartmouth.edu

Department of Mathematics
Dartmouth College

Math 23 Differential Equations
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Outline

1. Review of Power Series
   - Series
   - Power Series

2. Series Solutions
   - Motivating Example
   - Solutions Near Ordinary Points, Part 1
   - Solutions Near Ordinary Points, Part 2

3. Euler Equations & Regular Singular points
   - Real, Distinct Roots
   - Equal Roots
   - Complex Roots
   - Regular Singular Points
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The Definition

The expression

\[ \sum_{j=0}^{\infty} a_j, \]

where the \( a_j \)'s are real (or complex numbers) is called a **series**. For each \( N = 1, 2, 3, \ldots \) the expression

\[ S_N = \sum_{j=0}^{N} a_j = a_0 + a_1 + \cdots + a_N \]

is called the \( N \)-th partial sum of the series.
Definition

The series \( \sum_{j=0}^{\infty} a_j \), is said to converge if

\[
\lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{j=0}^{N} a_j
\]

exists. Otherwise we say the series diverges.
The series $\sum_{j=0}^{\infty} \frac{1}{2^j}$ converges to 2:

- $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$
- $S_n + \frac{1}{2^{n+1}} = S_{n+1} = 1 + \frac{1}{2}S_n$
- Solving for $S_n$ we get

$$S_n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

Therefore,

$$\lim_{n \to \infty} S_n = 2.$$
The series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges:

- $S_1 = 1$
- $S_2 = \frac{3}{2}$
- $S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq \frac{4}{2}$
- In general
  \[ S_{2^k} \geq \frac{k + 2}{2}. \]
- Therefore
  \[ \lim_{n \to \infty} S_n = \infty \]
Convergence Tests: The Comparison Test

**Theorem**

Suppose that $\sum_{j=0}^{\infty} a_j$ is a convergent series where $a_j \geq 0$ for all $j$. If $\{b_j\}_{j=1}^{\infty}$ is a sequence of numbers such that $|b_j| \leq a_j$ for all $j$, then the series $\sum_{j=0}^{\infty} b_j$ converges.
Convergence Tests: The Comparison Test

The series \( \sum_{j=0}^{\infty} \frac{\sin(j)}{2^j} \) converges:

- We recall that \( \sum_{j=0}^{\infty} \frac{1}{2^j} = 2 \).
- \( \left| \frac{\sin(j)}{2^j} \right| \leq \frac{1}{2^j} \) for all \( j \).
- Hence by the **Comparison Test** the series \( \sum_{j=0}^{\infty} \frac{\sin(j)}{2^j} \) converges.
Convergence Tests: The Ratio Test

**Theorem**

Consider a series $\sum_{j=0}^{\infty} a_j$ of non-zero terms. If

$$\lim_{j \to \infty} \frac{|a_{j+1}|}{|a_j|} < 1,$$

then the series converges.
Convergence Tests: The Ratio Test

The series $\sum_{j=1}^{\infty} \frac{2^j}{j!}$ converges:

- $a_j = \frac{2^j}{j!}$
- $\lim_{j \to \infty} \frac{|a_{j+1}|}{|a_j|} = \frac{2}{j+1} = 0$
- Therefore by the **Ratio Test**, the series converges.
Convergence Tests: The Alternating Series Test

**Theorem**

Let \( \{b_j\}_{j=1}^{\infty} \) be a sequence of nonnegative numbers such that

1. \( b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0; \)
2. \( \lim_{j \to \infty} b_j = 0. \)

Then the series

\[
\sum_{j=1}^{\infty} (-1)^j b_j
\]

converges.
The series \( \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \) converges:

- let \( b_j = \frac{1}{j} \)
- then \( b_j \geq b_{j+1} \geq 0 \) and \( \lim_{j \to \infty} b_j = 0 \)
- Therefore by the Alternating Series Test the series converges.
The expression

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to be a power series expanded about $x_0$. For each $N = 1, 2, 3, \cdots$ the expression

$$S_N(x) = \sum_{j=0}^{N} a_j (x - x_0)^j$$

is said to be the $N$-th partial sum of the Power series.
The power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) is said to

1. converge at \( x \) if

\[
\lim_{N \to \infty} S_N(x) = \lim_{N \to \infty} \sum_{j=0}^{N} a_j (x - x_0)^j
\]

exists.

2. converge absolutely at \( x \) if the series \( \sum_{n=0}^{\infty} |a_n| |x - x_0|^n \) converges at \( x \); that is, \( \lim_{N \to \infty} \sum_{n=0}^{N} |a_n| |x - x_0|^n \) exists.
Proposition

If the series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges absolutely at \( x \), then it converges. The converse need not be true.

Example

The power series \( \sum_{j=0}^{\infty} \frac{(-1)^j}{j} x^j \) converges at \( x = 1 \) (by the alternating series test), but it does not converge absolutely at \( x = 1 \) since the harmonic series

\[
\sum_{j=1}^{\infty} \frac{1}{j}
\]

diverges.
Interval of Convergence

**Proposition**

Assume \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges for \( x = c \). Then the power series converges for all \( x \) such that

\[
|x - x_0| < r = |c - x_0|.
\]

Hence, the set

\[
\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ converges} \}
\]

is an interval centered at \( x_0 \).
Definition

The radius of convergence $\rho$ of the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is

$$\rho = \text{Max}\{r : \sum_{n=0}^{\infty} a_n(x - x_0)^n \text{ converges for all } |x - x_0| < r\}.$$
Real Analytic

Definition

A function $f: U \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be real analytic if for each $x_0 \in U$, $f(x)$ may be represented by a convergent power series on an interval $I \subset U$ of positive radius centered at $x_0$:

$$f(x) = \sum a_n(x - x_0)^n.$$
Let \( f(x) = \sum a_n(x - x_0)^n \) and \( g(x) = \sum b_n(x - x_0)^n \) be power series centered at \( x_0 \) which converge on intervals \( I_1 \) and \( I_2 \) containing \( x_0 \) (resp.). Then on \( I_1 \cap I_2 \) we have

1. \( f(x) \pm g(x) = \sum (a_n \pm b_n)(x - x_0)^n \)
2. \( f(x)g(x) = \sum_{m=0}^{\infty} \sum_{j+k=m} (a_j b_k)(x - x_0)^m \).
The Ratio Test

**Theorem (Ratio Test)**

Consider the power series \( \sum_{n=0}^{\infty} a_n(x - x_0)^n \) and assume \( \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| \) exists and equals \( L \). Then the power series

1. **converges** for \( x \) such that \( |x - x_0|L < 1 \),
2. **diverges** for \( x \) such that \( |x - x_0|L > 1 \), and
3. for \( x \) such that \( |x - x_0|L = 1 \) we don’t know.
The Ratio Test: Examples

1. \( \cos(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \) converges for all \( x \).

2. \[ \sum_{n=1}^{\infty} \frac{n^2}{2^n} (x - 3)^n \] converges for all \( x \) such that \( |x - 3| < 2 \) and diverges for \( |x - 3| > 2 \). Need to check by hand the case \( |x - 3| = 2 \).

3. \[ \sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2 2^n} \] converges absolutely for \( |x + 1| < 2 \) and diverges for \( |x + 1| > 2 \). Need to check the case \( |x + 1| = 2 \) by hand.
1. Consider the series \( \sum_{n=k}^{\infty} a_n x^n \)

2. Make the substitution \( m = n - k \)

3. Then

\[
\sum_{n=k}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+k} x^{m+k} = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}
\]
Write the following series so that the generic term involves $x^n$

1. $\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2}$.
2. $\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$.
3. $\sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$. 
Definition

Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a power series.

1. The derived series is $\sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$

2. The integrated series is $\sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1}$

Theorem

The derived and integrated series have the same radius of convergence as $\sum_{n=0}^{\infty} a_n(x - x_0)^n$. 
Theorem

Let $f(x)$ be a real analytic function defined on an open interval $I$. Then $f$ is continuous and has continuous, real analytic derivatives of all orders. In fact, the derivatives of $f$ are obtained by differentiating its series representation term by term.
Corollary

Let $f$ be represented by a convergent power series on an interval of positive radius centered at $x_0$

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$
Examples

1. Find the radius of convergence of the following power series.
   a) \( \sum_{n=0}^{\infty} \frac{n}{2^n} x^n \)
   b) \( \sum_{n=0}^{\infty} \frac{(2x+1)^n}{n^2} \)

2. Find the Taylor Series of \( f(x) = \frac{1}{1-x} \) at \( x_0 = 0 \).
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Example

1. Consider the differential equation \( y'' + y = 0 \).
2. Assume that \( y(x) = \sum_{n=0}^{\infty} a_n x^n \).
3. We obtain the recurrence relation
   \[
   a_{2k} = (-1)^k \frac{a_0}{(2k)!} \quad \text{and} \quad a_{2k+1} = (-1)^k \frac{a_1}{(2k + 1)!}.
   \]
4. Then
   \[
   y(x) = a_0 \sum_{k} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k} (-1)^k \frac{x^{2k+1}}{(2k + 1)!} = a_0 \cos(x) + a_1 \sin(x)
   \]
Consider $P(x)y'' + Q(x)y' + R(x)y = 0$ (where $P, Q, R$ are polynomials with no common factors.)

Suppose $P(x_0) \neq 0$, then $x_0$ is called an ordinary point. Otherwise we say $x_0$ is singular.

Then on some interval $I$ containing $x_0$ we can write the ODE as

$$y'' + p(x)y' + q(x)y = 0.$$ 

Assume $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ and converges for $|x - x_0| < \rho$.

Substitute $y$, $y'$ and $y''$ into ODE and try to find a recurrence relation for the $a_n$'s. (This will require us to write the rational functions $p$ and $q$ as power series centered at $x_0$.)
Airy’s Equation: Series Solution at $x_0 = 0$

Find a power series solution to $y'' - xy = 0$ in a neighborhood of $x = 0$.

1. Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$
2. $y''' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$.
3. Then since $y'' - xy = 0$ we get

\[
0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x \sum_{n=0}^{\infty} a_n x^n
\]

\[
= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^n + 1
\]

\[
= 2a_2 + \sum_{n=1}^{\infty} ((n+2)(n+1)a_{n+2} - a_{n-1})x^n
\]
Airy’s Equation: Series Solution at $x_0 = 0$

We then conclude

- $a_2 = 0$
- we have the general recurrence relation

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad n \geq 1.$$  

Which implies that for $n \geq 1$

$$a_{3n} = \frac{a_0}{(2 \cdot 3)(5 \cdot 6) \cdots (3n - 1)(3n)}$$
$$a_{3n+1} = \frac{a_1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n)(3n+1)}$$
$$a_{3n+2} = a_2 = 0$$
Airy’s Equation: Series Solution at $x_0 = 0$

It then follows that our solution $y(x)$ has a Taylor series expansion of the form:

$$y(x) = \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2} x^{3n+2}$$

$$= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1}$$

$$= a_0 \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{(2 \cdot 3)(5 \cdot 6) \cdots (3n - 1)(3n)} x^{3n} \right\}$$

$$+ a_1 \left\{ x + \sum_{n=1}^{\infty} \frac{1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n)(3n+1)} x^{3n+1} \right\}$$
Setting

\[ y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{(2\cdot3)(5\cdot6)\cdots(3n-1)(3n)} x^{3n} \]
\[ y_2(x) = x + \sum_{n=1}^{\infty} \frac{1}{(3\cdot4)(6\cdot7)\cdots(3n)(3n+1)} x^{3n+1} \]

We can conclude that \( y_1 \) and \( y_2 \) are analytic functions which

1. Have an infinite radius of convergence (why?)
2. Solve Airy’s Equation on \(-\infty < x < \infty\) (why?)
3. Form a fundamental set of solutions on \(-\infty < x < \infty\) (why?). Hence, the general solution to Airy’s equation is

\[ y(x) = c_1 y_1(x) + c_2 y_2(x). \]
Airy’s Equation: Series Solution at $x_0 = 1$

Find a power series solution to $y'' - xy = 0$ in a neighborhood of $x = 1$.

1. Assume $y(x) = \sum a_n(x - 1)^n$
2. $y'' = \sum_{n=0}^\infty (n + 2)(n + 1)a_{n+2}(x - 1)^n$.
3. Then since $y'' - xy = 0$ we have

\[
0 = \sum_{n=0}^\infty (n + 2)(n + 1)a_{n+2}(x - 1)^n - x \sum_{n=0}^\infty a_n(x - 1)^n
\]

\[
= \sum_{n=0}^\infty (n + 2)(n + 1)a_{n+2}(x - 1)^n - (1 + (x - 1)) \sum_{n=0}^\infty a_n(x - 1)^n
\]
Airy’s Equation: Series Solution at $x_0 = 1$

Continuing we have

$$0 = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}(x - 1)^n$$

$$- \left\{ a_0 + \sum_{n=1}^{\infty} (a_n + a_{n-1})(x - 1)^n \right\}$$

$$= (2a_2 - a_0) + \sum_{n=1}^{\infty} ((n + 2)(n + 1)a_{n+2} - a_n - a_{n-1})(x - 1)^n$$
From which we deduce:

- \( a_2 = \frac{a_0}{2} \)
- \( a_{n+2} = \frac{a_n + a_{n-1}}{(n+2)(n+1)} \quad n \geq 1 \)

And we get

\[
y(x) = a_0 \left\{ 1 + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{6} + \frac{(x - 1)^4}{24} + \frac{(x - 1)^5}{30} + \cdots \right\} + a_1 \left\{ (x - 1) + \frac{(x - 1)^3}{6} + \frac{(x - 1)^4}{12} + \frac{(x - 1)^5}{120} + \cdots \right\}
\]

Hard to figure out a closed form formula for the \( a_n \)'s.
Setting

- \( y_1(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots \)
- \( y_2(x) = (x - 1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots \)

We’d like to be able to say what the radius of convergence of \( y_1 \) and \( y_2 \) is. However, since we cannot get a closed formula for the \( a_n \)’s we cannot do this directly via the ratio test. We will see in the next part that we will be able to estimate the radius of convergence . . .
Ordinary Point Revisited

Question
Do we really need to assume that $P$, $Q$ and $R$ are polynomials?

Definition
$x_0$ is said to be an ordinary point of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if the functions $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{Q(x)}$ are analytic at $x_0$. Otherwise, we say that $x_0$ is a singular point.
Ordinary Point Revisited

**Theorem**

If $x_0$ is an ordinary point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

then the general solution is of the form

$$y = \sum a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x),$$

where $a_0$ and $a_1$ are arbitrary and $y_1$ and $y_2$ are linearly independent series solutions centered at $x_0$. The radius of convergence of the Taylor series of the $y_i$’s centered at $x_0$ is at least as large as the minimum of the radii of convergence of the Taylor series of $p$ and $q$ centered at $x_0$. 
Ordinary Point Revisited

The solutions $y_1$ and $y_2$ in the previous theorem will be of the form:

$$y_1(x) = 1 + 0(x - x_0) + b_2(x - x_0)^2 + b_3(x - x_0)^3 + \cdots$$

and

$$y_2(x) = 0 + 1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots$$

- $y_1$ corresponds to the initial conditions
  $$y(x_0) = 1, \quad y'(x_0) = 0$$
- $y_2$ corresponds to the initial conditions
  $$y(x_0) = 0, \quad y'(x_0) = 1$$
Proposition

Consider the rational function \( h(x) = \frac{Q(x)}{P(x)} \), where \( P \) and \( Q \) are polynomials that do not have common factors. Then \( h \) is analytic at \( x_0 \) if and only if \( P(x_0) \neq 0 \). In the event that \( h \) is analytic at \( x_0 \), then the radius of convergence \( \rho \) of the Taylor series expansion of \( h \) at \( x_0 \) is given by

\[
\rho = \min \{|x_0 - r_1|, \ldots, |x_0 - r_k|\},
\]

where \( r_1, \ldots, r_k \) are the roots of \( P(x) \).

Remark

The polynomial \( P(x) \) might have complex roots, in which case the distance computed above is the distance in the complex plane.
Consider the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, where $P$, $Q$ and $R$ be polynomials (without common factors). If $x_0$ is a regular point of this ODE and $r_1, \ldots, r_k$ are the roots of $P(x)$, then the general solution of the ODE on an interval containing $x_0$ is of the form

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where $y_1$ and $y_2$ are analytic functions at $x_0$ and the radii of convergence of their respective Taylor series centered at $x_0$ are larger than $\min\{|x_0 - r_1|, \ldots, |x_0 - r_k|\}$. 
A Useful fact

Moral

The previous corollary allows us to estimate the radius of convergence (and hence the interval of solution) of our ODE without explicitly calculating the $a_n$’s and using the ratio test. Indeed, recall Airy’s equation

$$y'' - xy = 0.$$  

This has an analytic solution $y(x) = \sum_{n=0}^{\infty} a_n(x - 1)^n$ at $x_0 = 1$, but we previously noticed it was not possible to find a closed formula for the $a_n$’s. However, since $p(x) = 0$ and $q(x) = -x = -1 - (x - 1)$ both have infinite radii of convergence at $x_0 = 1$, we see that the analytic solution $y(x)$ centered at $x_0 = 1$ will also have infinite radius of convergence.
Examples

Determine a lower bound for the radius of convergence of the series solution of

\[(x^2 - 2x - 3)y'' + xy' + 4y = 0\]

centered at

1. \(x_0 = 4\);
2. \(x_0 = 0\);
3. \(x_0 = -4\).
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Examples

Question
How do we analyze/solve 2nd order ODEs near singular points?
Euler’s Equation

- Consider the homogeneous ODE
  \[ L[y] = x^2 y'' + \alpha xy' + \beta y = 0 \]
- Has a singularity at \( x = 0 \)
- Suppose the solution is of the form \( y = x^r \equiv e^{r \ln(x)} \)
- Then we get \( L[x^r] = 0 \) if and only if
  \[ F(r) = r(r - 1) + \alpha r + \beta = 0 \]
- But, \( F(r) = (r - r_1)(r - r_2) \), where
  \[ r_1, r_2 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2} \]
Euler's Equation: Real, Distinct Roots

- if \( r_1 \neq r_2 \) are real, then

\[
W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2+1}.
\]

does not vanish for \( x > 0 \)

- Hence \( \{x^{r_1}, x^{r_2}\} \) is a fundamental set of solutions to the ODE on \( x > 0 \).
Example: Euler’s Equation with Real, Distinct Roots

Solve the initial value problem

\[ 2x^2 y'' + 3xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 2, \quad x > 0 \]

- the General solution to ODE on \( x > 0 \) is

\[ y(x) = c_1 x^{1/2} + c_2 x^{-1}. \]

- On \( x > 0 \) the IVP is satisfied by

\[ y(x) = 2x^{1/2} - x^{-1}. \]
The Derivation: Equal Roots

- Recall that $x^r = e^{r \ln(x)}$, so $\frac{\partial}{\partial r} x^r = x^r \ln(x)$.
- Suppose $r_1 = r_2$, then $F(r) = (r - r_1)^2$.
- $\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r} (x^r F(r))$

\[
L[x^r \ln(x)] = L\left[ \frac{\partial}{\partial r} x^r \right] \\
= \frac{\partial}{\partial r} L[x^r] \\
= x^r \ln(x)(r - r_1)^2 + 2(r - r_1)x^r \\
= 0 \text{ (if } r = r_1)\\
\]
The Derivation: Equal Roots

Hence, \( x^{r_1} \ln(x) \) is a solution.

\[ W(x^{r_1}, x^{r_1} \ln(x)) = x^{2r_1 - 1} > 0 \text{ for } x > 0 \]

Hence \( \{x^{r_1}, x^{r_1} \ln(x)\} \) forms a fundamental set of solutions.

The general solution in this case is

\[
(c_1 + c_2 \ln(x))x^{r_1}.
\]
An Example: Euler’s Equation with Equal Roots

Solve the following 2nd order IVP

\[ x^2 y'' + 5xy' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 0, \quad x > 0 \]

- The general solution to ODE on \( x > 0 \) is
  \[ y(x) = x^{-2}(c_1 + c_2 \ln(x)). \]

- On \( x > 0 \) the IVP is satisfied by
  \[ y(x) = x^{-2}(2 + 2 \ln(x)) = 2x^{-2}(1 + \ln(x)). \]
The Derivation: Complex Roots

- $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$

- So for $x > 0$

  $x^{r_1} = e^{(\lambda + i\mu)\ln(x)}$
  
  $= e^{\lambda\ln(x) + i\mu\ln(x)}$
  
  $= e^{\lambda\ln(x)}e^{i\mu\ln(x)}$
  
  $= x^\lambda(\cos(\mu\ln(x)) + i\sin(\mu\ln(x)))$

and

  $x^{r_2} = x^\lambda(\cos(\mu\ln(x)) - i\sin(\mu\ln(x)))$. 
The Derivation: Complex Roots

- \( \{x^{r_1}, x^{r_2}\} \) forms a Fundamental set of solutions.
- \( \{x^\lambda \cos(\mu \ln(x)), x^\lambda \sin(\mu \ln(x))\} \) is a fundamental set of solutions consisting of real-valued functions.
- So the general solution is of the form

\[
c_1 x^\lambda \cos(\mu \ln(x)) + c_2 x^\lambda \sin(\mu \ln(x)), \quad x > 0.
\]
Example: Euler’s Equation with Complex Roots

Solve the IVP

\[ x^2y'' + xy' + y = 0, \quad y(1) = 0, \quad y'(1) = 3, \quad x > 0 \]

- the General solution to ODE on \( x > 0 \) is

\[ y(x) = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)). \]

- On \( x > 0 \) the IVP is satisfied by

\[ y(x) = 3 \sin(\ln(x)). \]
What about $x < 0$?

- In the three previous cases we restricted to the interval $x > 0$.
- On the interval $x < 0$ we get

$$y(x) = \begin{cases}  
  c_1|x|^{r_1} + c_2|x_2|^{r_2} \\
  (c_1 + c_2 \ln(|x|))|x|^{r_1} \\
  c_1|x|^{\lambda} \cos(\mu \ln(|x|)) + c_2|x|^{\lambda} \sin(\mu \ln(|x|))
\end{cases}$$

Depending on the roots $r_1, r_2$ of $F(r) = r(r - 1) + \alpha r + \beta$. 
The Definition

Consider the second order ODE of the form

\[ P(x)y'' + Q(x)y' + R(x)y = 0. \]

and let \( x_0 \) be a point where \( P(x_0) = 0 \). \( x_0 \) is said to be a regular singular point if

\[ \lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \]

are both finite. Otherwise, we say \( x_0 \) is an irregular singular point.
The singularity at $x_0 = 0$ of Euler’s equation is regular.

In general one can handle regular singular points in manner analogous to what we did for Euler’s equation.

We won’t cover this, but it is useful in studying Bessel’s equation.
1. Classify the singular points of the following ODE

\[ 2x(x - 2)^2 y'' + 3xy' + (x - 2)y = 0. \]

2. Find the general solution to the following ODE that is valid on any interval not containing the singular point.

\[ (x - 1)^2 y'' + 8(x - 1)y' + 12y = 0 \]