Math 24 Exam 2 - Take Home portion

Directions: Your solutions to these exam problems are due at the beginning of class on Monday, 26 February 2001. You may feel free to use your book or your class notes from this course to help you with these problems, but **you may not use any other source of information** This includes books, papers, people and electronic devices. The only exception is that you may ask your instructor for clarification.

Give complete and careful solutions to the problems. **Justify all assertions and conclusions**. You would be wise to quote any theorem you use by number or page number to make precise your reference (e.g., by Theorem 1.9 or by the Corollary on page 50...). Also, if you cannot prove one part of a problem, you may assume its truth to help you prove the next part of the problem. Last, but not least, neatness counts!

4. (10) Let A be the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 6 & 2 \\ 1 & 0 & 2 & 4 & 3 \end{pmatrix}$, and $\mathbf{b}$ the column vector $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ 13 \\ 14 \end{pmatrix}$.

   (a) Find the rank of A.

   (b) Find all solutions to the system of linear equations $A\mathbf{x} = \mathbf{0}$.

   (c) Find all solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$.

   Hint: It may be more efficient to consider these questions in reverse order.

5. (10) Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, and let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation defined by $T(B) = AB$.

   (a) Find the matrix of T with respect to the ordered basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

   (b) Find the rank and nullity of T.

   (c) Find the range of T.

6. (10) Let V be the vector space defined by $V = \{A \in M_{3\times 3}(\mathbb{R}) \mid \text{Trace}(A) = 0\}$. You may assume that V is a vector space. Show that V is isomorphic to $P_7(\mathbb{R})$. **Note:** It is not always necessary to construct an isomorphism to show two vector spaces are isomorphic.

7. (10) Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose range is spanned by the vectors $(1, 2, 3, 4)$, $(5, 6, 7, 8)$, and $(7, 10, 13, 16)$, and with nullspace spanned by $(1, 3, -2)$? If so, describe how you would construct all such transformations. If not, explain why it is not possible. How would the answer change if the vector $(7, 10, 13, 16)$ were replaced by the vector $(7, 10, 13, 15)$?

8. (10) Let V be a two-dimensional vector space over a field F and let $T : V \to V$ be a linear map for which $T^2 = 0$, but $T \neq 0$ (that is, T^2 is the zero transformation, but not T).

   (a) Show that the range and nullspace of T each have dimension one.

   (b) Show that there is a basis $\mathcal{B}$ of V so that $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

   where $\lambda \in F$ and $\lambda \neq 0$.