First some definitions. If $W_1$ and $W_2$ are two subspaces of $V$, we define

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1 \& w_2 \in W_2 \}.$$ 

In other words, $W_1 + W_2$ is the collection of all vectors you can get by adding an element of $W_1$ to an element of $W_2$. If $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$, then we say $V$ is the *direct sum* of $W_1$ and $W_2$, and we write $V = W_1 \oplus W_2$.

1. Prove that $W_1 + W_2$ is the smallest subspace containing both $W_1$ and $W_2$. (In other words, $W_1 + W_2$ is the span of $W_1 \cup W_2$.)
2. Give examples of pairs of subspaces $W_1$ and $W_2$ of $\mathbb{R}^3$, neither of which is contained in the other, such that:

   (a) $W_1 + W_2 \neq \mathbb{R}^3$. In your example, what is $W_1 + W_2$?

   (b) $W_1 + W_2 = \mathbb{R}^3$, but $\mathbb{R}^3$ is not the direct sum of $W_1$ and $W_2$. In your example, what is $W_1 \cap W_2$?

   (c) $\mathbb{R}^3$ is the direct sum of $W_1$ and $W_2$.

3. Suppose $W_1$ and $W_2$ are both subspaces of a finite-dimensional vector space $V$. Make a conjecture about the relationship among the dimensions of $W_1$, $W_2$, $W_1 \cap W_2$, and $W_1 + W_2$. 
4. Express $M_{2\times 2}(\mathbb{C})$ as the direct sum of two nonzero subspaces.

5. Express $P(\mathbb{R})$ as the direct sum of two nonzero subspaces in two ways.
   
   (a) One of the subspaces has finite dimension.

   (b) Both of the subspaces are infinite-dimensional.
6. Prove the conjecture you made in problem (3). Hint: A basis \{x_1, \ldots, x_k\} for \(W_1 \cap W_2\) can be extended to a basis \{x_1, \ldots, x_k, y_1, \ldots, y_n\} for \(W_1\). It can also be extended to a basis \{x_1, \ldots, x_k, z_1, \ldots, z_m\} for \(W_2\). For homework, you might want to verify your conjecture by looking at problem 29(a) of section 1.6 of the textbook. Please make a conjecture yourself first, though.
7. Every vector in $W_1 + W_2$ can be expressed as a sum, $w_1 + w_2$, of vectors $w_1 \in W_1$ and $w_2 \in W_2$. In what cases is this expression unique? Prove your answer is correct.