Math 24  
Spring 2012  
Problems from Monday April 9

First some definitions. If $W_1$ and $W_2$ are two subspaces of $V$, we define

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1 \& w_2 \in W_2 \}.$$ 

In other words, $W_1 + W_2$ is the collection of all vectors you can get by adding an element of $W_1$ to an element of $W_2$. If $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$, then we say $V$ is the direct sum of $W_1$ and $W_2$, and we write $V = W_1 \oplus W_2$.

1. Prove that $W_1 + W_2$ is the smallest subspace containing both $W_1$ and $W_2$. (In other words, $W_1 + W_2$ is the span of $W_1 \cup W_2$.)

To see $W_1 + W_2$ is a subspace, check closure under addition and under multiplication by scalars. Let $w_1 + w_2$ and $w'_1 + w'_2$ be any elements of $W_1 + W_2$, where $w_1, w'_1 \in W_1$ and $w_2, w'_2 \in W_2$, and let $a$ be any scalar. Then, since $W_1$ and $W_2$ are closed under addition and under multiplication by scalars,

$$(w_1 + w_2) + (w'_1 + w'_2) = (w_1 + w'_1) + (w_2 + w'_2) \in W_1 + W_2,$$

$$a(w_1 + w_2) = aw_1 + aw_2 \in W_1 + W_2.$$ 

Also, $W_1 \subseteq W_1 + W_2$, since every $w_1 \in W_1$ can be written $w_1 = w_1 + 0 \in W_1 + W_2$. For the same reason, $W_2 \subseteq W_1 + W_2$. We have shown $W_1 + W_2$ is a subspace containing both $W_1$ and $W_2$.

Clearly every element $w_1 + w_2$ of $W_1 + W_2$ is in span$(W_1 + W_2)$ so $W_1 + W_2 \subseteq$ span$(W_1 \cup W_2)$.

To show $W_1 + W_2 = \text{span}(W_1 \cup W_2)$, it remains only to show that span$(W_1 \cup W_2) \subseteq W_1 + W_2$. But this must be true, because we have shown $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$, and span$(W_1 \cup W_2)$ is the smallest such subspace.

We could also show span$(W_1 \cup W_2) \subseteq W_1 + W_2$ directly. Let $w \in \text{span}(W_1 + W_2)$. We can write $w$ as a linear combination of elements of $W_1 \cup W_2$,

$$w = a_1 u_1 + \cdots + a_n u_n + b_1 v_1 + \cdots b_m v_m,$$
where $u_i \in W_1$ and $v_j \in W_2$. But then, $a_1u_1 + \cdots + a_n u_n \in W_1$, and $b_1 v_1 + \cdots + b_m v_m \in W_2$, and

$$w = (a_1u_1 + \cdots + a_n u_n) + (b_1 v_1 + \cdots b_m v_m) \in W_1 + W_2,$$

so $\text{span}(W_1 \cup W_2) \subseteq W_1 + W_2$. 
2. Give examples of pairs of subspaces $W_1$ and $W_2$ of $\mathbb{R}^3$, neither of which is contained in the other, such that:

(a) $W_1 + W_2 \neq \mathbb{R}^3$. In your example, what is $W_1 + W_2$?

(b) $W_1 + W_2 = \mathbb{R}^3$, but $\mathbb{R}^3$ is not the direct sum of $W_1$ and $W_2$. In your example, what is $W_1 \cap W_2$?

(c) $\mathbb{R}^3$ is the direct sum of $W_1$ and $W_2$.

This is a homework problem.

3. Suppose $W_1$ and $W_2$ are both subspaces of a finite-dimensional vector space $V$. Make a conjecture about the relationship among the dimensions of $W_1$, $W_2$, $W_1 \cap W_2$, and $W_1 + W_2$.

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

Intuitively, we add up the number of dimensions in $W_1$ and $W_2$, and then subtract the number of dimensions in the overlap, because they were counted twice.
4. Express $M_{2 \times 2}(\mathbb{C})$ as the direct sum of two nonzero subspaces.

There are many possible answers. A straightforward one is:

$$W_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \quad W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \mid c, d \in \mathbb{C} \right\}$$

A possibly more interesting solution is to let $W_1$ be the subspace of matrices with zero trace, and $W_2$ be the subspace of diagonal matrices whose two diagonal entries are equal. It’s easy to see their intersection contains only the zero matrix. You can see that together they span the entire space by writing down simple bases for $W_1$ and $W_2$,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

and observing that from them you can generate all the standard basis elements.

5. Express $P(\mathbb{R})$ as the direct sum of two nonzero subspaces in two ways.

(a) One of the subspaces has finite dimension.

(b) Both of the subspaces are infinite-dimensional.

This is a homework problem.
6. Prove the conjecture you made in problem (3). Hint: A basis \( \{x_1, \ldots, x_k\} \) for \( W_1 \cap W_2 \) can be extended to a basis \( \{x_1, \ldots, x_k, y_1, \ldots, y_n\} \) for \( W_1 \). It can also be extended to a basis \( \{x_1, \ldots, x_k, z_1, \ldots, z_m\} \) for \( W_2 \). For homework, you might want to verify your conjecture by looking at problem 29(a) of section 1.6 of the textbook. Please make a conjecture yourself first, though.

This is a challenging homework problem.
7. Every vector in $W_1 + W_2$ can be expressed as a sum, $w_1 + w_2$, of vectors $w_1 \in W_1$ and $w_2 \in W_2$. In what cases is this expression unique? Prove your answer is correct.

This answer is unique just in case $W_1 \cap W_2 = \{0\}$; that is, just in case the sum $W_1 + W_2$ is a direct sum.

To show this, first suppose $W_1 + W_2 \neq \{0\}$, and let $w$ be a nonzero element of $W_1 \cap W_2$. Then $w$ can be expressed as a sum of a vector from $W_1$ and a vector from $W_2$ in more than one way, namely as $w + 0$ and as $0 + w$.

Conversely, suppose that $W_1 \cap W_2 = \{0\}$. We must show any vector $w \in W_1 + W_2$ can be expressed as a sum of a vector from $W_1$ and a vector from $W_2$ in only one way. To do this, suppose we have two such expressions $w = w_1 + w_2$ and $w = w_1' + w_2'$. We must show $w_1 = w_1'$ and $w_2 = w_2'$.

We have $w_1 + w_2 = w_1' + w_2'$, which we can rewrite as $w_1 - w_1' = w_2' - w_2$. Thus $w_1 - w_1'$ is in both $W_1$ and $W_2$. The only vector in both $W_1$ and $W_2$ is $0$, so $w_1 - w_1' = 0$, and $w_1 = w_1'$. The same argument shows that $w_2 = w_2'$. 
