The space $P_2(\mathbb{R})$ has for standard ordered basis $\alpha = \{1, x, x^2\}$ and another ordered basis
\[ \beta = \{\frac{1}{2}x^2 - \frac{1}{2}x, 1-x^2, \frac{1}{2}x^2 + \frac{1}{2}x\} . \]
The space $\mathbb{R}^3$ has the standard ordered basis $\gamma = \{e_1, e_2, e_3\}$.
Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by $T(f) = (f(-1), f(0), f(1))$ for every $f(x) \in P_2(\mathbb{R})$.

For example,
\[ T(x^2 + x + 1) = \begin{pmatrix} (-1)^2 + (-1) + 1 \\ 0^2 + (0) + 1 \\ (1)^2 + (1) + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} . \]

(a) Compute $[x^2]_{\beta}$, $[x]_{\beta}$, $[1]_{\beta}$. Justify your answers.

**Solution.** Since
\[ x^2 = 1(\frac{1}{2}x^2 - \frac{1}{2}x) + 0(1-x^2) + 1(\frac{1}{2}x^2 + \frac{1}{2}x), \]
we see that $[x^2]_{\beta} = (1, 0, 1)$.

Since
\[ x = -1(\frac{1}{2}x^2 - \frac{1}{2}x) + 0(1-x^2) + 1(\frac{1}{2}x^2 + \frac{1}{2}x), \]
we see that $[x]_{\beta} = (-1, 0, 1)$.

Since
\[ 1 = 1(\frac{1}{2}x^2 - \frac{1}{2}x) + 1(1-x^2) + 1(\frac{1}{2}x^2 + \frac{1}{2}x), \]
we see that $[1]_{\beta} = (1, 1, 1)$.

(b) Compute $[T]_{\gamma}^{\alpha}$ and $[T]_{\beta}^{\gamma}$. Justify your answers.

**Solution.** Since $T(1) = (1, 1, 1)$, $T(x) = (-1, 0, 1)$, $T(x^2) = (1, 0, 1)$, we see that
\[ [T]_{\gamma}^{\alpha} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} . \]

Since $T(\frac{1}{2}x^2 - \frac{1}{2}x) = (1, 0, 0)$, $T(1-x^2) = (0, 1, 0)$, $T(\frac{1}{2}x^2 + \frac{1}{2}x) = (0, 0, 1)$, we see that
\[ [T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \]

(c) Show that $[T]_{\gamma}^{\alpha}[f]_{\alpha} = [f]_{\beta}$ for every $f(x) \in P_2(\mathbb{R})$. 

Solution. By Theorem 2.14, we have 
\[
[T(f)]_\gamma = [T]_\alpha^\gamma [f]_\alpha \quad \text{and} \quad [T(f)]_\gamma = [T]_\beta^\gamma [f]_\beta.
\]
Therefore, 
\[
[T]_\alpha^\gamma [f]_\alpha = [T]_\beta^\gamma [f]_\beta.
\]
Since \([T]_\beta^\gamma\) is the 3 \times 3 identity matrix by part (b), we also have \([T]_\beta^\gamma [f]_\beta = [f]_\beta\). Therefore, \([T]_\alpha^\gamma [f]_\alpha = [f]_\beta.\) □

Solution. Suppose \(f(x) = a + bx + cx^2\). By part (a), we see that 
\[
[f]_\beta = [a1 + bx + cx^2]_\beta = a[1]_\beta + b[x]_\beta + c[x^2]_\beta = \begin{pmatrix} a - b + c \\ a \\ a + b + c \end{pmatrix}.
\]
On the other hand, 
\[
[T]_\alpha^\gamma [f]_\alpha = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - b + c \\ a \\ a + b + c \end{pmatrix}.
\]
Therefore, \([T]_\alpha^\gamma [f]_\alpha = [f]_\beta\). □