The Rank of a Matrix

Lecture 16

February 14, 2007
The Rank of a Matrix

Definition

Let $A \in M_{m \times n}(F)$. The rank of $A$, denoted $\text{rank}(A)$ is defined to be the rank of the linear transformation $L_A : F^n \rightarrow F^m$. 
An \( n \times n \) matrix is invertible if and only if its rank is \( n \).

If \( T : V \to W \) is a linear transformation and \( \beta \) and \( \gamma \) are ordered bases for \( V \) and \( W \), then \( \text{rank}(T) = \text{rank}([T]_{\gamma}^{\beta}) \).
Fact

- An $n \times n$ matrix is invertible if and only if its rank is $n$. 
Properties

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- An $n \times n$ matrix is invertible if and only if its rank is $n$.
- If $T : V \rightarrow W$ is a linear transformation and $\beta$ and $\gamma$ are ordered bases for $V$ and $W$, then $\text{rank}(T) = \text{rank}([T]^{\gamma}_{\beta})$. 
Theorem

Let $A$ be an $m \times n$ matrix. If $P$ and $Q$ are invertible $m \times m$ and $n \times n$ matrices, respectively, then

1. $\text{rank}(AQ) = \text{rank}(A) = \text{rank}(PA)$.
2. $\text{rank}(PAQ) = \text{rank}(A)$.
3. Elementary row and column operations on a matrix are rank-preserving.
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2. \( \text{rank}(PAQ) = \text{rank}(A) \).
3. Elementary row and column operations on a matrix are rank-preserving.
Theorem

The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.
How to Compute the Rank of a Matrix?

**Theorem**

Let $A$ be an $m \times n$ matrix of rank $r$. Then $r \leq n$ and $r \leq m$, and, by means of a finite number of elementary row and columns operations, $A$ can be transformed into the matrix

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix},$$

where $0_1, 0_2, \text{ and } 0_3$ are zero matrices.
Corollary

Let $A$ be an $m \times n$ matrix of rank $r$. Then there exist invertible matrices $B$ and $C$ of sizes $m \times m$ and $n \times n$, respectively, such that $D = BAC$, where

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}.$$
Corollary

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Let $A$ be an $m \times n$ matrix. Then

1. $\text{rank}(A^t) = \text{rank}(A)$.

2. The rank of any matrix equals the maximum number of its linearly independent rows.
Consequences

Corollary

Let $A$ be an $m \times n$ matrix. Then

1. $\text{rank}(A^t) = \text{rank}(A)$.
2. The rank of any matrix equals the maximum number of its linearly independent rows.
3. The rows and columns of any matrix generate subspaces of the same dimension, equal to the rank of the matrix.
Corollary

Every invertible matrix is a product of elementary matrices.