Math 31: Maximal Ideals

Halloween

Let R be a commutative ring with multiplicative identity 1. A proper ideal $M \subsetneq R$ is a **maximal ideal** if for any ideal I with $M \subset I \subset R$, either $I = M$ or $I = R$. Maximal ideals some of the nicest ideals in a ring. The point of these exercises is to investigate maximal ideals.

1. Let I and J be ideals of R. Show that $I \subset I + J \subset R$. In the same vein, $J \subset I + J \subset R$.

2. Let I and J be ideals of R. Show that $I + J$ is the smallest ideal containing both I and J as subsets.

3. Find the maximal ideals of $\mathbb{Z}$.

Hint: Here is one strategy to show an ideal $M \subset R$ is maximal. Suppose for some ideal I, we have $M \subsetneq I \subset R$. To show M is maximal, we have to prove $I = R$. Because $M \subsetneq I$, there exists some $r \in I$ not contained in M. Then we know $M + (r) \subset I \subset R$. If we can show $M + (r) = R$, then we must have $M + (r) = I = R$, which is what we wanted to show.

4. Suppose M is a maximal ideal of R. Show that R/M is a field.

Hint: We know that the quotient ring R/M is a commutative ring with multiplicative identity $1 + M$. Suppose $x + M$ is a nonzero element of R/M. Using the fact that M is maximal, show that $x + M$ has a multiplicative inverse.

5. Show that the converse of the previous problem is true as well. That is, if $M \subset R$ is an ideal and R/M is a field, then M is a maximal ideal.