Math 31: Q is for Quaternion

Wednesday October 10 2012

On October 16, 1843, Hamilton and his wife took a walk through the city of Dublin. Suddenly he found what he'd been seeking. He took out his carving implement, and carved into a stone bridge the group multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Let Q denote the group of quaternions. This group of order 8 consists of the elements $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. We can think of Q as an extension of the group of complex fourth roots of unity $\theta_4 = \{1, -1, i, -i\}$. The element $1 \in Q$ is the group identity, and the self-inverse element $-1 \in Q$ also commutes with everything. The other six elements $\pm i$, $\pm j$, and $\pm k$ may not.

With your neighbors, investigate the group Q through following questions.

1.) Using Hamilton's multiplication rules, compute the six products below.

ij $\qquad jk$ $\qquad ki$ $\qquad ji$ $\qquad kj$ $\qquad ik$

2.) Make a subgroup lattice for Q. You may want to start by finding the cyclic subgroups.

3.) If G is an arbitrary group, we know that the center

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$$

is a subgroup of G. Prove that $Z(G) \lhd G$.

4.) What is $Z(Q)$?

5.) Prove the following interesting property of Q: Every subgroup of Q is normal.

6.) We know of four nonisomorphic groups of order 8: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_8$, and D_4. Because Q is nonabelian, it cannot be isomorphic to any of the first three. Is Q isomorphic to D_4? Or does Q make a fifth isomorphism class of groups of order 8? Why?