Math 31: Symmetries of the Tetrahedron

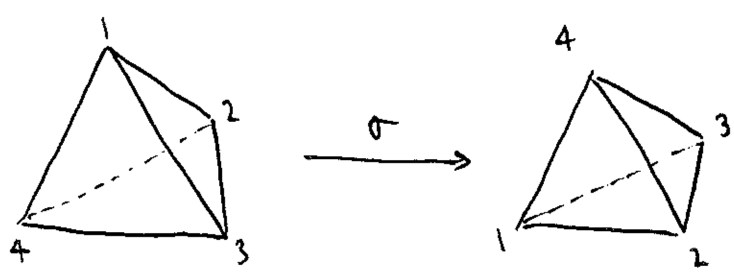
Tuesday October 2 2012

Let T be the group of symmetries which you are able to perform on the tetrahedron. There are additional symmetries (such as reflection through a mirror), but you would not be able to perform such motions without tearing the tetrahedron and resembling it, or venturing into the 4th dimension.

We can write an the element of T in a more concrete way. Any symmetry of the tetrahedron must take a vertex to another vertex. Let us label the vertices of the tetrahedron with the numbers 1, 2, 3, and 4 (perhaps you want to let the numbers 1, 2, 3, and 4 correspond to different color beads on your tetrahedron). If $\sigma \in T$ is a symmetry of the tetrahedron, then σ gives rise to permutation

$$\tilde{\sigma} : \{1,2,3,4\} \longrightarrow \{1,2,3,4\}$$

where σ sends the vertex labelled i to the vertex labelled $\tilde{\sigma}(i)$. Here is an example:



$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

From the permutation $\tilde{\sigma}$, we can completely recover σ, since any symmetry of the tetrahedron can be described by what it does to the vertices. In this way, via the association $\sigma \longleftrightarrow \tilde{\sigma}$, we can think of T as a subgroup of S_4. On the other hand, not every element in S_4 gives rise to a symmetry of the tetrahedron. For example, you won't find a symmetry σ of the tetrahedron with

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

Try it out.

Investigate the group T (with your neighbors) by pondering the following questions.

1.) What are the elements of T? In other words, which permutations in S_4 give rise to a symmetry of the tetrahedron? What is the order of T?

2.) For each element $\sigma \in T$, find its order.

3.) Is T cyclic? Is T abelian?

4.) Find the center $Z(T)$.

5.) Find as many subgroups of T as you can. Draw a lattice. Are there any subgroups of order 6?

6. An *isomorphism* is a bijective homomorphism. Pick a subgroup H of T and find an isomorphism between H and a group we've previously studied.