Proof by Contrapositive

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So far we've practiced some different techniques for writing proofs. We started with **direct proofs**, and then we moved on to proofs by **contradiction** and **mathematical induction**. The method of contradiction is an example of an **indirect proof**: one tries to skirt around the problem and find a clever argument that produces a logical contradiction. This is not the only way to perform an indirect proof - there is another technique called **proof by contrapositive**.

Suppose that we are asked to prove a **conditional statement**, or a statement of the form

$$\text{“If } A\text{, then } B\text{.”}$$

We know that we can try to prove it directly, which is always the more enlightening and preferred method. If a direct proof fails (or is too hard), we can try a contradiction proof, where we assume $\neg B$ and A, and we arrive at some sort of fallacy. It's also possible to try a proof by contrapositive, which rests on the fact that a statement of the form

$$\text{“If } A\text{, then } B\text{.” } (A \implies B)$$

is logically equivalent to

$$\text{“If } \neg B\text{, then } \neg A\text{.” } (\neg B \implies \neg A)$$

The second statement is called the **contrapositive** of the first. Instead of proving that A implies B, you prove directly that $\neg B$ implies $\neg A$.

Proof by contrapositive: To prove a statement of the form "If A, then B," do the following:

1. Form the contrapositive. In particular, negate A and B.
2. Prove directly that $\neg B$ implies $\neg A$.

There is one small caveat here. Since proof by contrapositive involves negating certain logical statements, one has to be careful. If the statements are at all complicated, negation can be quite delicate. However, sometimes the given proposition already contains certain negative statements, and contrapositive is the natural choice.

Example 1. Prove by contrapositive: Let $a, b, n \in \mathbb{Z}$.If $n \nmid ab$, then $n \nmid a$ and $n \nmid b$.

Proof. We need to find the contrapositive of the given statement. First we need to negate "$n \nmid a$ and $n \nmid b$." This is an example of a case where one has to be careful, the negation is

$$\text{“}n \mid a \textbf{ or } n \mid b\text{.”}$$

(The "and" becomes an "or" because of DeMorgan's law.) The initial hypothesis is easy to negate: $n \mid ab$. Therefore, we are trying to prove

$$\text{“If } n \mid a \text{ or } n \mid b\text{, then } n \mid ab\text{.”}$$

Suppose that n divides a. Then $a = nc$ for some $c \in \mathbb{Z}$, and

$$ab = ncb = n(cb),$$

so $n \mid ab$. Similarly, if $n \mid b$, then $b = nd$ for some $d \in \mathbb{Z}$, and

$$ab = and = n(ad),$$

so $n \mid ab$. Therefore, we have proven the result by contraposition. □

Here's another example. In this one, a direct proof would be awkward (and quite difficult), so contrapositive is the way to go.

Example 2. Prove by contrapositive: Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then x is odd.

Proof. Suppose that x is even. Then we want to show that $x^2 - 6x + 5$ is odd. Write $x = 2a$ for some $a \in \mathbb{Z}$, and plug in:

$$\begin{aligned} x^2 - 6x + 5 &= (2a)^2 - 6(2a) + 5 \\ &= 4a^2 - 12a + 5 \\ &= 2(2a^2 - 6a + 2) + 1. \end{aligned}$$

Thus $x^2 - 6x + 5$ is odd. □

Tips and Tricks for Proofs

The four proof techniques that we've talked about are really the only ones that people ever use. However, there are some general tips regarding the types of statements that you may be asked to prove. We've already seen some of these via example in class, but we'll list them here anyway.

If and only if: Sometimes you are asked to prove something of the form "A if and only if B" or "A is equivalent to B." The usual way to do this is to prove two things: first, prove that "A implies B," and then prove that "B implies A." Use any of the possible techniques to prove these two implications.

Uniqueness: You are often asked to prove that some object satisfying a given property is unique. We've seen before that the standard trick is to assume that there is another object satisfying the property, and then show that it actually equals the original one.

Existence: This sort of proof often goes hand in hand with uniqueness. You are given some specified property, and then asked to show that an object exists which has that property. There are often two ways to do this. One can offer up a **constructive** proof, in which the object is explicitly constructed. A nonconstructive proof is the exact opposite - it shows that the object exists, but it gives no indication as to what the object looks like.

Existence and Uniqueness: As stated before, existence and uniqueness go hand in hand. Sometimes you will be asked to prove that an object satisfying some property exists and is unique. This can be done in either order. Sometimes it is easy to prove that the object exists, and then to show that it is unique. However, the existence proof may seem daunting, and it is often helpful to prove uniqueness first. The uniqueness proof may give some hints as to what the object must look like.