Math 31 – Homework 6

Due Monday, August 12 (Changed from August 9)

Note: Any problem labeled as "show" or "prove" should be written up as a formal proof, using complete sentences to convey your ideas.

Easier

1. We will see in class that the kernel of any homomorphism is a normal subgroup. Conversely, you will show that any normal subgroup is the kernel of some homomorphism. That is, let G be a group with N a normal subgroup of G, and define a function $\pi : G \to G/N$ by

$$\pi(g) = Ng$$

for all $g \in G$. Prove that π is a homomorphism, and that $\ker \pi = N$.

2. Recall that $\mathbb{R}^\times$ is the group of nonzero real numbers (under multiplication), and let $N = \{-1, 1\}$. Show that N is a normal subgroup of $\mathbb{R}^\times$, and that $\mathbb{R}^\times/N$ is isomorphic to the group of positive real numbers under multiplication. [**Hint:** Use the Fundamental Homomorphism Theorem.]

3. [Saracino, #13.1] Let $\varphi : \mathbb{Z}_8 \to \mathbb{Z}_4$ be given by

$$\varphi(x) = [x]_4,$$

i.e., the remainder of x mod 4. Find $\ker \varphi$. To which familiar group is $\mathbb{Z}_8/\ker \varphi$ isomorphic?

4. If G is a group and $M \trianglelefteq G$, $N \trianglelefteq G$, prove that $M \cap N \trianglelefteq G$. [You proved on an earlier assignment that $M \cap N$ is a subgroup of G, so you only need to prove that it is normal.]

5. Classify all abelian groups of order 600 up to isomorphism.

Medium

6. [Saracino, #11.17 and 11.18] Let G be a group, and let $H \leq G$.

(a) If G is abelian, prove that G/H is abelian. [**Hint:** You may want to use a result from the last homework assignment.]

(b) Prove that if G is cyclic, then G/H is also cyclic.

7. Prove that if G_1 and G_2 are abelian groups, then $G_1 \times G_2$ is abelian.

8. If G_1 and G_2 are groups, prove that $G_1 \times G_2 \cong G_2 \times G_1$.

9. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$, and let $H = \langle (1, 0) \rangle$. Find all the right cosets of H in G, and compute the quotient group G/H. (That is, identify it with a more familiar group.)