1. (4 points). Let $C[0,1]$ denote the set of all continuous functions $f(x), x \in [0,1]$ with the scalar product defined as $(f,g) = \int_0^1 f(x)g(x)dx$. Let $p_0(x) = 1$ be the null-degree polynomial on $x \in [0,1]$. Find the first-degree polynomial $p_1(x) = a + bx$ such that $p_0 \perp p_1$ and $\|p_1\| = 1$.

*Solution.* We find $a$ and $b$ from the conditions

$$(p_0, p_1) = \int_0^1 (a + bx)dx = 0, \quad \|p_1\|^2 = \int_0^1 (a + bx)^2dx = 1.$$ 

We have

$$\int_0^1 (a + bx)dx = a + b/2 = 0,$$

$$\int_0^1 (a + bx)^2dx = \int_0^1 (a^2 + 2abx + b^2x^2)dx = a^2 + ab + b^2/3 = 1.$$

From the first equation we have $b = -2a$ and substituting it into the second equation we obtain $a^2 - 2a^2 + 4a^2/3 = 1$ which yields $a = \pm \sqrt{3}$ and $b = -2\sqrt{3}$ if $a = \sqrt{3}$ and $b = 2\sqrt{3}$ if $a = -\sqrt{3}$ (two solutions). Finally, we have $p_1(x) = \pm \sqrt{3}(1 - 2x)$.

2. (5 points). Check that functions $p_0(x) = 1$ and $p_1(x) = x - \pi/4$ are orthogonal on $C[0,\pi/2]$. Find the best linear approximation of $f(x) = \sin x$ by $p_0$ and $p_1$ on $[0,\pi/2]$. Compute the squared norm of approximation.

*Solution.* To prove the orthogonality we need to show that $\int_0^{\pi/2} (x - \pi/4)dx = 0$. We have

$$\int_0^{\pi/2} (x - \pi/4)dx = \frac{1}{2} \left( \frac{\pi}{2} \right)^2 - \frac{\pi}{2} \frac{\pi}{4} = 0.$$ 

The best linear approximation for $f(x)$ is found in the form $\lambda_0 p_0(x) + \lambda_1 p_1(x)$ where

$$\lambda_0 = \frac{(f,p_0)}{\|p_0\|^2}, \quad \lambda_1 = \frac{(f,p_1)}{\|p_1\|^2}.$$ 

We have

$$\begin{align*}
(f,p_0) &= \int_0^{\pi/2} \sin xdx = - \cos x \bigg|_0^{\pi/2} = 1, \quad \|p_0\|^2 = \int_0^{\pi/2} dx = \pi/2, \\
(f,p_1) &= \int_0^{\pi/2} (x - \pi/4) \sin xdx = \int_0^{\pi/2} x \sin xdx - \pi/4 \int_0^{\pi/2} \sin xdx = 1 - \pi/4, \\
\|p_1\|^2 &= \int_0^{\pi/2} (x - \pi/4)^2dx = \pi^3/96.
\end{align*}$$
Thus,

\[ \lambda_0 = \frac{2}{\pi}, \lambda_1 = \frac{1 - \pi/4}{\pi^3/96}, \]

and the best linear approximation for \( \sin x \) on \( [0, \pi/2] \) is

\[ \hat{f} = \lambda_0 p_0(x) + \lambda_1 p_1(x) = \frac{2}{\pi} + 96 \frac{1 - \pi/4}{\pi^3}(x - \frac{\pi}{4}) = 0.1477 + 0.66444x, \]

see the graph below. The squared norm of approximation is computed as

\[
\int_0^{\pi/2} (\sin x - 0.1477 - 0.66444x)^2dx = \int_0^{\pi/2} \sin^2 x dx - \int_0^{\pi/2} (0.1477 + 0.66444x)^2 dx = 7.8919 \times 10^{-3}.
\]

3. (5 points). Find the first-order approximation of \( \sin x \) at \( x_0 = 0 \) using Taylor series expansion and compute the squared norm of approximation using the scalar product \( (f, g) = \int_0^{\pi/2} f(x)g(x)dx \).

Solution. Taylor series expansion of the first-order of \( \sin x \) at \( x_0 = 0 \) gives \( \sin x \simeq x \). The squared norm of approximation is

\[
\int_0^{\pi/2} (x - \sin x)^2dx = \int_0^{\pi/2} \sin^2 x dx - 2 \int_0^{\pi/2} x \sin x dx + \int_0^{\pi/2} x^2 dx = \frac{1}{4}\pi - 2 + \frac{1}{24}\pi^3 = 7.7326 \times 10^{-2}
\]

We notice that the error is larger than in the previous problem. It must be larger because the approximation in the previous problem provides the minimum of the error.

Approximation of \( \sin x \) on \( [0, \pi/2] \) by two linear functions. The first is the best linear approximation (solid) and the second is the first-order Taylor series expansion (dashed). The latter has a larger approximation error.

4. (7 points). Determine the Fourier series expansion of the function \( f(x) = \pi^2 - x^2 \) for \( -\pi \leq x \leq \pi \). Compute the squared norm of approximation based on the first two terms of the Fourier series.
Solution. Since \( f(x) \) is an even function \( b_n = 0 \). The constant term is

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) dx = \frac{4}{3} \pi^2.
\]

The \( a_n \) term is

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) \cos nx \, dx = \pi \int_{-\pi}^{\pi} \cos nx \, dx - \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx
\]

\[
= \pi \left( \frac{1}{n} \sin nx \right) \bigg|_{-\pi}^{\pi} - \frac{1}{\pi} \left( \frac{2x \cos nx}{n^2} + \frac{(n^2 x^2 - 2) \sin nx}{n^3} \right) \bigg|_{-\pi}^{\pi}
\]

\[
= (-1)^{n+1} \frac{4}{n^2}.
\]

Finally, the Fourier series is

\[
\pi^2 - x^2 = \frac{2}{3} \pi^2 + 4 \left( \cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \frac{1}{4^2} \cos 4x + \ldots \right)
\]

The first two terms give the approximation

\[
\pi^2 - x^2 \approx \frac{2}{3} \pi^2 + 4 \cos x,
\]

plotted below.

![Plot of \( \pi^2 - x^2 \) on \( [-\pi, \pi] \), solid line approximates based on the first two terms of Fourier series.]

The squared norm of approximation is computed by the formula

\[
\int_{-\pi}^{\pi} (\pi^2 - x^2 - \frac{2}{3} \pi^2 - 4 \cos x)^2 \, dx
\]

\[
= \int_{-\pi}^{\pi} (\pi^2 - x^2)^2 \, dx - \pi \left( \frac{1}{2} a_0^2 + a_1^2 \right) = \frac{16}{15} \pi^5 - \pi \left( \frac{16}{18} \pi^4 + 4^2 \right) = 4.138.
\]