1. (p. 26, 2, 5 points). What is your interpretation of the initial-boundary-value problem?
PDE: \( u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad t > 0 \); BCs: \( u(0,t) = 0, u_x(1,t) = 1, \quad t > 0 \); IC: \( u(x,0) = \sin(\pi x), \quad 0 \leq x \leq 1 \). Can you draw rough sketches of the solution for different values of time? Will the solution come to a steady state; is this obvious?

Solution. The right boundary condition \( u_x(1,t) = 1, \quad t > 0 \) means that the heat flows in with unit speed. Since the steady state solution \( \overline{u} = \overline{u}(x) \) for PDE \( u_t = \alpha^2 u_{xx} \) is a linear function of \( x \) it implies that \( \overline{u}(x) = x \). Indeed, \( \overline{u}(0) = 0 \) and \( \overline{u}_x(1) = 1 \). The sketch of the temperature distribution at different time is shown below. Notice, the bold line has slope 1 at \( x = 1 \).

![Temperature Distribution Sketch](image)

Dashed line – initial condition temperature \((t = 0)\), solid – steady state solution \((t = \infty)\), bold line – temperature for intermediate time.

2. (p. 26, 4, 5 points). Suppose a metal rod laterally insulated has an initial temperature of 20°C but immediately thereafter has one end fixed at 50°C. The rest of the rod is immersed in a liquid solution temperature 30°C. What would be the IBVP that describes this problem.

Solution. If \( L \) is the length of the rod than IC is \( u(x,0) = 20, \quad 0 \leq x \leq L \). Let the temperature of the right end is kept at 50°C that implies BC \( u(L,t) = 50 \) and for the left end we have \( u(0, t) = 30 \).
for all time $t > 0$.

3. (p. 31, 1, 5 points). Substitute the units of each quantity $u, u_t, ...$ into the equation $u_t = \alpha^2 u_{xx} - \beta u$ to see that every term has the same units of $^\circ C/\text{sec}$.

Solution. $u_t$ measures the speed at which temperature changes with time, i.e. the units are $^\circ C/\text{sec}$. Let us assume the length of the rod is measured in cm. Then $u_{xx}$ is measured in $^\circ C/cm^2$ and $\alpha^2$ has units $cm^2/sec$ so that $\alpha^2 u_{xx}$ is measured in $cm^2/sec \times ^\circ C/cm^2 = ^\circ C/sec$. Parameter $\beta$ must be measured in $1/sec$ in order to have $^\circ C/sec$ for the last term, $\beta u$.

4. (p. 42, 5, 5 points). What is the solution to problem 4 if the IC is changed to

$$u(x,0) = \sin(2\pi x) + \frac{1}{3} \sin(4\pi x) + \frac{1}{5} \sin(6\pi x).$$

Solution. The key point to the solution is the Fourier sine expansion of $\phi(x) = u(x,0)$. But since $u(x,0)$ is a linear combination of sine functions we can determine the Fourier coefficients without evaluating integrals, or more precisely, $n = 1, A_1 = 0, n = 2, A_2 = 1, n = 3, A_3 = 0, n = 4, A_4 = 1/3, n = 5, A_5 = 0, n = 6, A_6 = 1/5,$ and $A_n = 0$ for $n > 6$. Therefore, the solution is ($\alpha = 1$)

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(\alpha n)^2 t} \sin(\pi n x)$$

$$= e^{-4\pi^2 t} \sin(2\pi x) + \frac{1}{3} e^{-16\pi^2 t} \sin(4\pi x) + \frac{1}{5} e^{-36\pi^2 t} \sin(6\pi x).$$