6. Assume that $g\colon \mathbf{R} \to \mathbf{R}$ is an odd function, or a function defined only for $x \geq 0$. Use the Fourier transform pair

$$f(x) = \int_{-\infty}^{\infty} F(s)e^{2\pi isx}\,ds$$

where

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx$$

to derive the Fourier sine transform pair

$$g(x) = 2\int_{0}^{\infty} G(s)\sin(2\pi sx)\,ds$$

where

$$G(s) = 2\int_{0}^{\infty} g(x)\sin(2\pi sx)\,dx.$$

7. Find the Fourier sine transform $\mathcal{F}_{\sin}(g)(s) = G(s)$ where

$$g(x) = e^{-x}, \quad x > 0.$$

To save you from another integration by parts, you may use

$$\int e^{-x}\sin cx\,dx = \frac{-e^{-x}}{1+c^2}(c\cos cx + \sin cx).$$

answer: $\mathcal{F}_{\sin}(e^{-x})(\omega) = \frac{2\pi s}{1+(2\pi s)^2}$

8. Solve the following heat conduction problem for a semi-infinite rod.

$$4u_{xx} = u_t, \quad x > 0, \quad t > 0$$

$$u(0,t) = 0, \quad t \geq 0$$

$$u(x,0) = e^{-x}, \quad x > 0.$$

Your answer may contain an integral.

9. Find the Fourier transform $U(s,t)$ of the solution to the inhomogenous heat equation

$$\alpha^2 u_{xx} + f(x,t) = u_t, \quad -\infty < x < \infty, \quad t > 0$$

which satisfies the initial condition

$$u(x,0) = 0, \quad -\infty < x < \infty.$$

Then write an expression for the solution $u(x,t)$ itself. You will need to remember (from Math 8 or Math 23) how to solve a first order linear ODE.

10.

(a) Use Parseval's identity to evaluate the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi s}{\pi s}\right)^2 ds.$$

(b) Use the inverse Fourier transform (evaluated at 0) to compute

$$\int_{-\infty}^{\infty} \frac{\sin \pi s}{\pi s}\, ds.$$

(c) Then change variables and use symmetry to show that

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(This definite integral came up in our discussion of the Gibbs' Phenomenon.)