6. Assume that \( g: \mathbb{R} \to \mathbb{R} \) is an odd function, or a function defined only for \( x \geq 0 \). Use the Fourier transform pair

\[
f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i s x} \, ds
\]

where

\[
F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} \, dx
\]

to derive the Fourier sine transform pair

\[
g(x) = 2 \int_{0}^{\infty} G(s) \sin(2\pi s x) \, ds
\]

where

\[
G(s) = 2 \int_{0}^{\infty} g(x) \sin(2\pi s x) \, dx.
\]

7. Find the Fourier sine transform \( \mathcal{F}_{\sin}(g)(s) = G(s) \) where

\[g(x) = e^{-x}, \quad x > 0.\]

To save you from another integration by parts, you may use

\[
\int e^{-x} \sin cx \, dx = \frac{e^{-x}}{1 + c^2} (c \cos cx + \sin cx).
\]

\text{answer:} \quad \mathcal{F}_{\sin}(e^{-x})(\omega) = \frac{2\pi s}{1 + (2\pi s)^2}

8. Solve the following heat conduction problem for a semi-infinite rod.

\[
4u_{xx} = u_t, \quad x > 0, \quad t > 0
\]

\[
u(0,t) = 0, \quad t \geq 0
\]

\[
u(x,0) = e^{-x}, \quad x > 0.
\]

Your answer may contain an integral.

9. Find the Fourier transform \( U(s,t) \) of the solution to the inhomogenous heat equation

\[
\alpha^2 u_{xx} + f(x,t) = u_t, \quad -\infty < x < \infty, \quad t > 0
\]

which satisfies the initial condition

\[
u(x,0) = 0, \quad -\infty < x < \infty.
\]
Then write an expression for the solution $u(x,t)$ itself. You will need to remember (from Math 8 or Math 23) how to solve a first order linear ODE.

10. 
(a) Use Parseval’s identity to evaluate the integral 
\[ \int_{-\infty}^{\infty} \left( \frac{\sin \pi s}{\pi s} \right)^2 ds. \]

(b) Use the inverse Fourier transform (evaluated at 0) to compute 
\[ \int_{-\infty}^{\infty} \frac{\sin \pi s}{\pi s} ds. \]

(c) Then change variables and use symmetry to show that 
\[ \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}. \]

(This definite integral came up in our discussion of the Gibbs’ Phenomenon.)