Quiz 2, Math 38, Spring 2012

Instructions:
Always (briefly) explain or demonstrate why, unless told otherwise.
In counting problems, answers like \( \binom{5}{2} + 2^4 + 3 \times 5 \) are usually preferable to "41".

(1) How many spanning trees does \( K_5 \) have?

\[
\text{Since } K_5 \text{ has every tree on 5 vertices, } K_5 \text{ has } n-2 = 5-2 = 3 \text{ sp. trees}
\]

(2) How many spanning trees does the following graph have?

\[
\tau(G) = \tau(G-e) + \tau(G+e)
\]

\[
= \tau(K) + \tau(K) = 5 + 2 \times 3 + 2 \times 2 + 3 \times 2 = 21
\]

(3) How many trees are there in the same isomorphism class as the one below?

\[
\text{One vertex of degree 4, one of degree 2 (uniquely identifies those vertices)}
\]

\[
\text{Then 3 vertices adjacent to the } d(v)=4 \text{ vertex that are leaves.}
\]

\[
6 \times 5 \times \binom{4}{3} = 120
\]
(4) For the following graph,
(a) label each vertex with its eccentricity,
(b) trace over the center of $G$,
(c) give its diameter,
(d) give its radius.

No need to justify.

\[ \text{diam} = 3 \]
\[ \text{rad} = 2 \]

(5) For the same graph as above, without drawing the complement $\bar{G}$, can you give upper and/or lower bounds on the diameter of $\bar{G}$?

Since $\text{diam}(\bar{G}) \leq 3$,
we have $\text{diam}(\bar{G}) \leq 3$.
Also $\bar{G}$ has edges, so
$\text{diam}(\bar{G}) \geq 2$. 
(6) Is the following statement true or false? If true, explain why. If false, give a counter-example and a similar statement which is true.

A graph G is a tree if and only if every edge is a cut edge.

False: e is not a tree, but e is a cut edge.

Better:
A graph G is a tree if and only if it is connected and every edge is cut.
Or:
A graph G is a forest if and every edge is cut.

(7) State four (other?) ways to characterize trees with n vertices.

(No need to prove, just state them. Don’t reuse (6).)

Some set of properties X characterize trees on n vertices if you can make the statement "A graph G with n vertices is a tree if and only if G has properties X".

(i) G is connected and acyclic
(ii) G is connected w/ n-1 edges
(iii) G is acyclic w/ n-1 edges
(iv) For every pair u, v ∈ V(G), there is exactly one u-v path in G, and G is loopless.
(v) Adding any edge w/ endpoints in V(G) creates exactly one cycle, and G is loopless.