Homework for Friday, October 27. Remember, class is at 9:15.

1. A universal ($\forall_1$) formula is one of the form $\forall x_1 \cdots \forall x_n \, \theta$, where θ is quantifier-free. An existential ($\exists_1$) formula is of the dual form $\exists x_1 \cdots \exists x_n \, \theta$. Let $\mathfrak{A}$ be a substructure of $\mathfrak{B}$, and let $s : V \to |\mathfrak{A}|$.

   (a) Show that if $\models_{\mathfrak{B}} \varphi[s]$ and φ is universal, then $\models_{\mathfrak{A}} \varphi[s]$. And if $\models_{\mathfrak{A}} \psi[s]$ and ψ is existential, then $\models_{\mathfrak{B}} \psi[s]$.

   (b) Conclude that the sentence $\forall x \, Px$ is not logically equivalent to any existential sentence, nor $\exists x \, Px$ to any universal sentence.

2. An $\exists_2$ formula is one of the form $\exists x_1 \cdots \exists x_n \, \theta$, where θ is universal.

   (a) Show that if an $\exists_2$ sentence not containing function symbols is true in $\mathfrak{A}$, then it is true in some finite substructure of $\mathfrak{A}$.

   (b) Conclude that $\forall x \, \exists y \, Pxy$ is not logically equivalent to any $\exists_2$ sentence.

3. Assume the language has equality and a two-place predicate symbol P. Consider the two structures $(\mathbb{N}, <)$ and $(\mathbb{R}, <)$ for the language. Find a sentence true in one structure and false in the other.

4. Let a and b be two distinct objects. Consider the set of all possible structures $\mathfrak{A}$ such that $|\mathfrak{A}| = \{a, b\}$ and $P^{\mathfrak{A}}$ is a binary relation on $\{a, b\}$. (We assume the language consists only of equality and P.) List all such structures, and determine which of them are isomorphic.