Homework for Friday, November 17.

1. Given two linear orderings $(A, <_A)$ and $(B, <_B)$ we may form an ordering $A + B$ as follows: Let the universe of $A + B$ consist of the set $\{\,(0, a) : a \in A\,\} \cup \{\,(1, b) : b \in B\,\}$ ordered by placing $(0, a) <' (1, b)$ for all a and b, $(0, a_1) <' (0, a_2)$ whenever $a_1 <_A a_2$, and $(1, b_1) <' (1, b_2)$ whenever $b_1 <_B b_2$.

   Now find a sentence φ that distinguishes $\mathbb{N} + \mathbb{N}$ from $(\mathbb{N}, <)$.

2. Find a sentence distinguishing $\mathbb{Z} + \mathbb{Z}$ from $(\mathbb{Z}, <)$.

3. Let m and n be two elements of bZ.

   (a) Show that there is an automorphism h of $(\mathbb{Z}, <)$ (i.e., an isomorphism of $(\mathbb{Z}, <)$ to $(\mathbb{Z}, <)$) such that $h(m) = n$.

   (b) Show that every automorphism of $(\mathbb{Z}, <)$ has this form.

   (c) Use the first two parts of this problem to conclude that there is no definition of 0 in $(\mathbb{Z}, <)$.

4. It is shown in the text that any two countable dense linear orderings without endpoints are isomorphic. Here $(A, <_A)$ is a dense linear ordering without endpoints if it satisfies the sentences

$$\forall x\, \forall y (x < y \lor x \approx y \lor y < x)$$
$$\forall x\, \forall y (x < y \to y \not< x)$$
$$\forall x\, \forall y\, \forall z (x < y \to (y < z \to x < z))$$
$$\forall x\, \forall y (x < y \to \exists z (x < z \land z < y))$$
$$\forall x\, \exists y\, \exists z (y < x \land x < z)$$

   (a) Using this information, show that if $(A, <_A)$ and $(B, <_B)$ are two countable dense linear orderings with endpoints (satisfying the first four axioms above and

$$\exists x\, \forall y (x < y \lor x \approx y) \land \exists x\, \forall y (y < x \lor y \approx x)$$

   then they are isomorphic.

   (b) Up to isomorphism, how many countable linear orderings are there there that satisfy the first four axioms?