Exponential Growth and Decay

10/31/2005

- Derivatives are functions that measure rates of change.

- A rate of change can be a powerful tool for expressing quantitatively a qualitative description.

- We know that a nation's population grows or declines depending on the birth and death rates.

- Does it make sense to say that at any time t, the rate of change of the size of a growing population is proportional to its size?

- Let $y(t)$ be the size of the population at time t.

$$\frac{dy}{dx} = ky; y(0) = y_0.$$

- Note that if $k > 0$, then the population is growing, and if $k < 0$, then the population is decreasing.

Theorem. *The IVP* $\frac{dy}{dt} = ky$, $y(0) = y0$, k *constant, has unique solution* $y = y^{ekt}$.

Example

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$$\begin{aligned} y(12) &= 700e^{12k} \\ 900 &= 700e^{12k} \\ \frac{900}{700} &= e^{12k} \\ \ln\left(\frac{900}{700}\right) &= 12k \\ k &= \frac{\ln 900 - \ln 700}{12} \end{aligned}$$

Doubling Time and Half-Life

- In an exponential growth model, the doubling time is the length of time required for the population to double.

- In a decay model, the half-life is the length of time required for the population to be reduced to half its size.

- A characteristic of exponential models is that these numbers are independent of the point in time from which the measurement begins.

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- To find k we use:

$$\frac{y_0}{2} = y^{600k}$$

Newton's Law of Cooling

- This law states that a hot object introduced into an environment maintained at a fixed cooler temperature will cool at a rate proportional to the difference between its own temperature and that of the surrounding environment.

- That is, if $y(t)$ is the temperature of the object t units of time after it is introduced into a medium at fixed temperature T_m, we have
$$\frac{dy}{dt} = k(y - T_m); y(0) = y_0$$
where k is a constant.

Example

- Suppose a metal object at 112 degrees Fahrenheit is removed from boiling water and placed on a plate in a room maintained at 68 degrees F.

- Suppose the object cools to 90 degrees in 5 minutes.

- How long will it take to cool to 80 degrees? Note that in this problem, $T_m = 68$, and $y_0 = 112$.

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- The solution is
$$y - T_m = (y_0 - T_m)e^{kt}$$
$k = (\ln 22 - \ln 44)/5.$

Separable Differential Equations

- A first-order differential equation in x and y is called *separable* if it is of the form
$$\frac{dy}{dx} = g(x)h(y).$$

- That is the x's and dx's can be put on one side of the equation and the y's and dy's on the other

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Example

- As a review, let's again solve the equation

$$\frac{dy}{dx} = ky$$

by the method of separation of variables.

Justification for the Method of Separation of Variables

- We need to show that given the equation

$$\frac{dy}{dx} = g(x)h(y)$$

the antiderivative of $\frac{1}{h(y)}$ as a function of y equals the antiderivative of $g(x)$ as a function of x.

$$\begin{aligned} f'(x) &= g(x)h(f(x)) \\ \frac{f'(x)}{h(f(x))} &= g(x) \end{aligned}$$

- Let $H(y)$ be any antiderivative of $1/h(y)$

$$\begin{aligned} \frac{d}{dx} H(f(x)) &= H'(f(x)) f'(x) \\ &= f'(x) \frac{1}{h(f(x))} \\ &= g(x) \end{aligned}$$

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- Thus, the solution $y = f(x)$ satisfies the equation

$$H(f(x)) = \int g(x)\mathsf{d}x$$

Examples

- Solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

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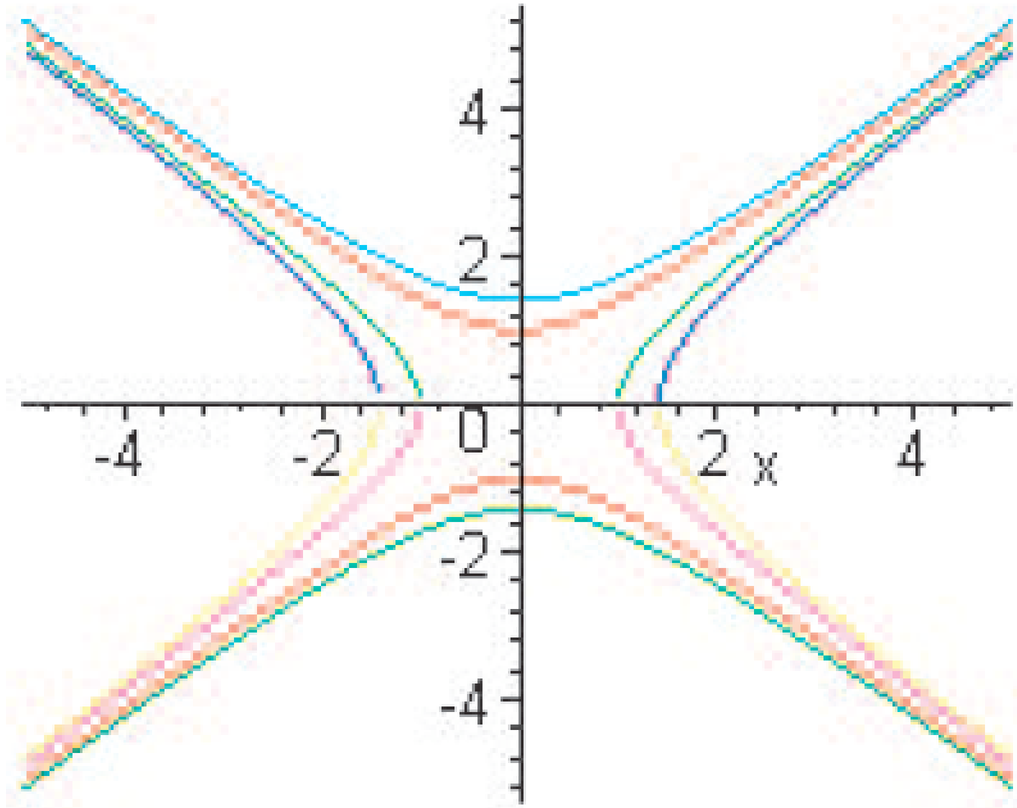
- The solution is

$$y^2 - x^2 = 1$$



4
2
-4
-2
0
2 x
4
-2
-4

- Solve the IVP

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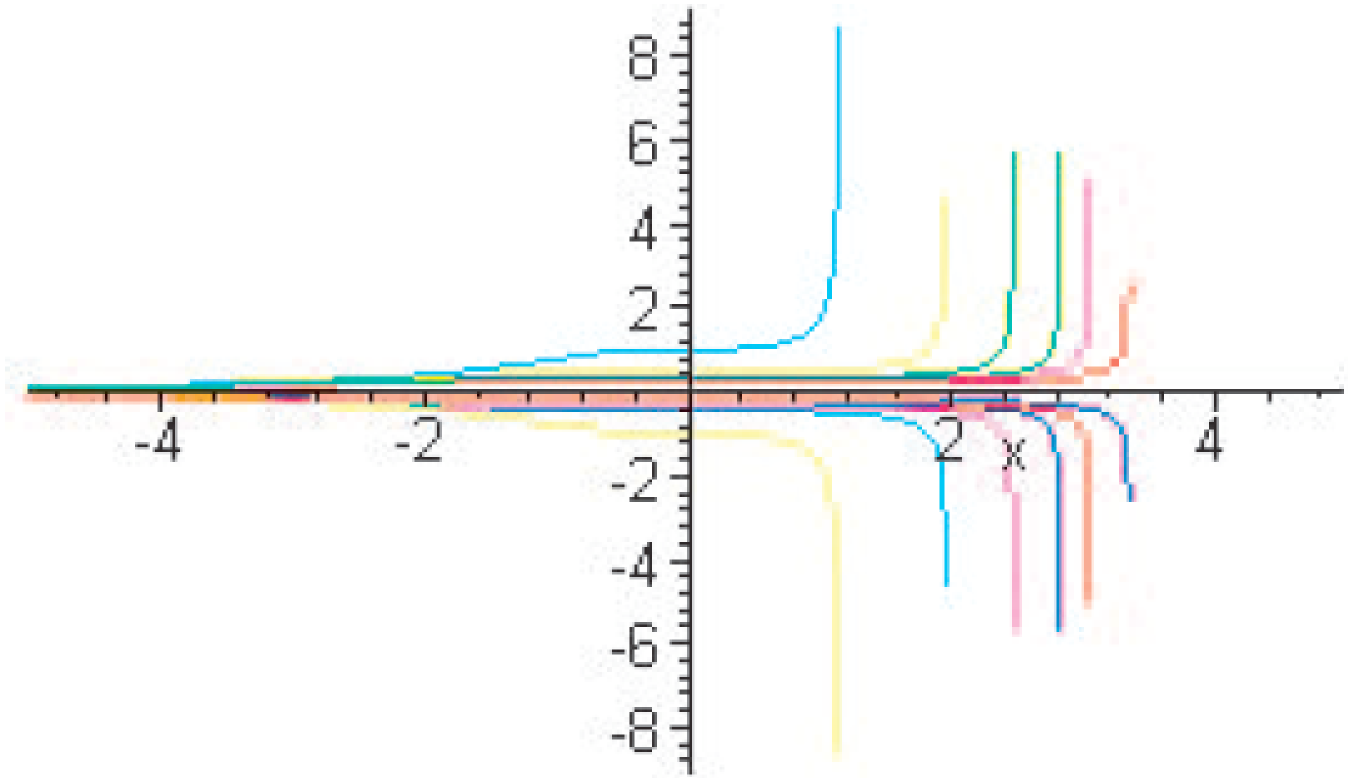
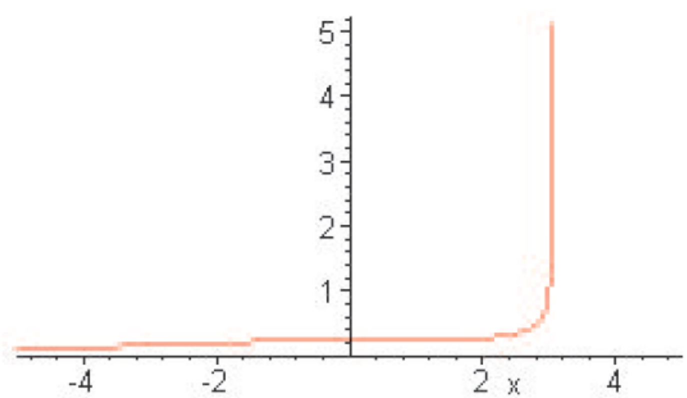
- The solution is

$$-\frac{1}{2y^2} = \frac{x^3}{3} + C.$$


8
6
4
2
-2
-4
-6
-8
-4
-2
2
x
4

- From $y(3) = 1$, we find the particular solution:

$$C = -\frac{19}{2}$$
$$y = \sqrt{\frac{1}{19 - \frac{x^3}{3}}}$$



Example

- Solve

$$\frac{dy}{dx} = \frac{2y}{x}.$$

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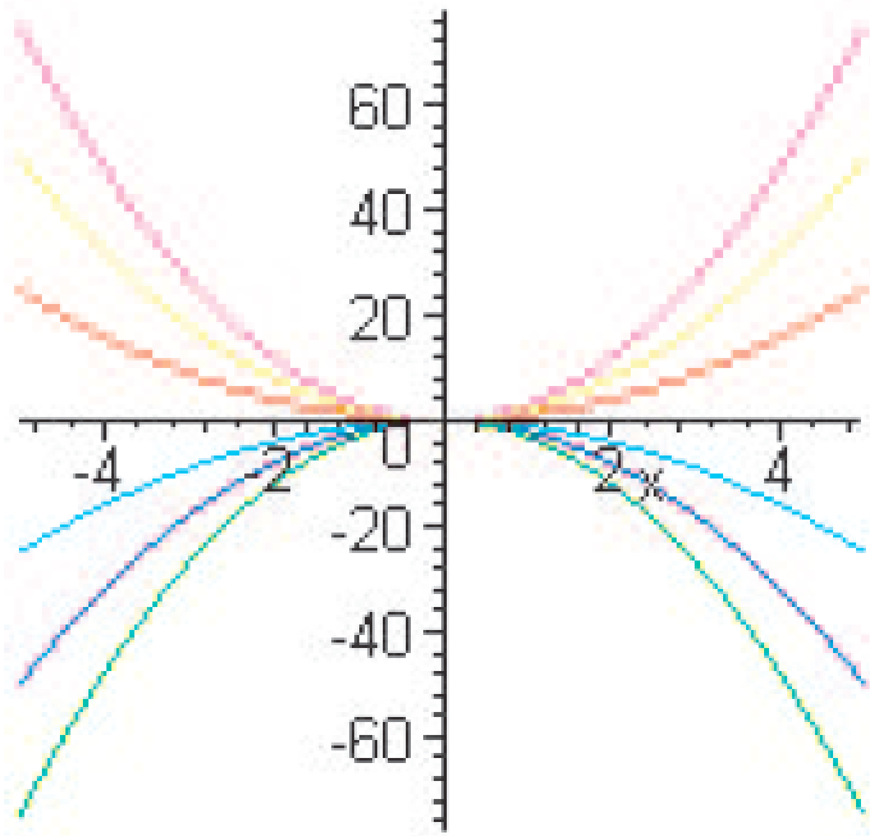
- The solution is

$$y = C_1 x^2$$



60
40
20
0
-20
-40
-60
-4
-2
2
x
4

Example

- Solve
$$\frac{dy}{dx} = -\frac{x}{2y}.$$

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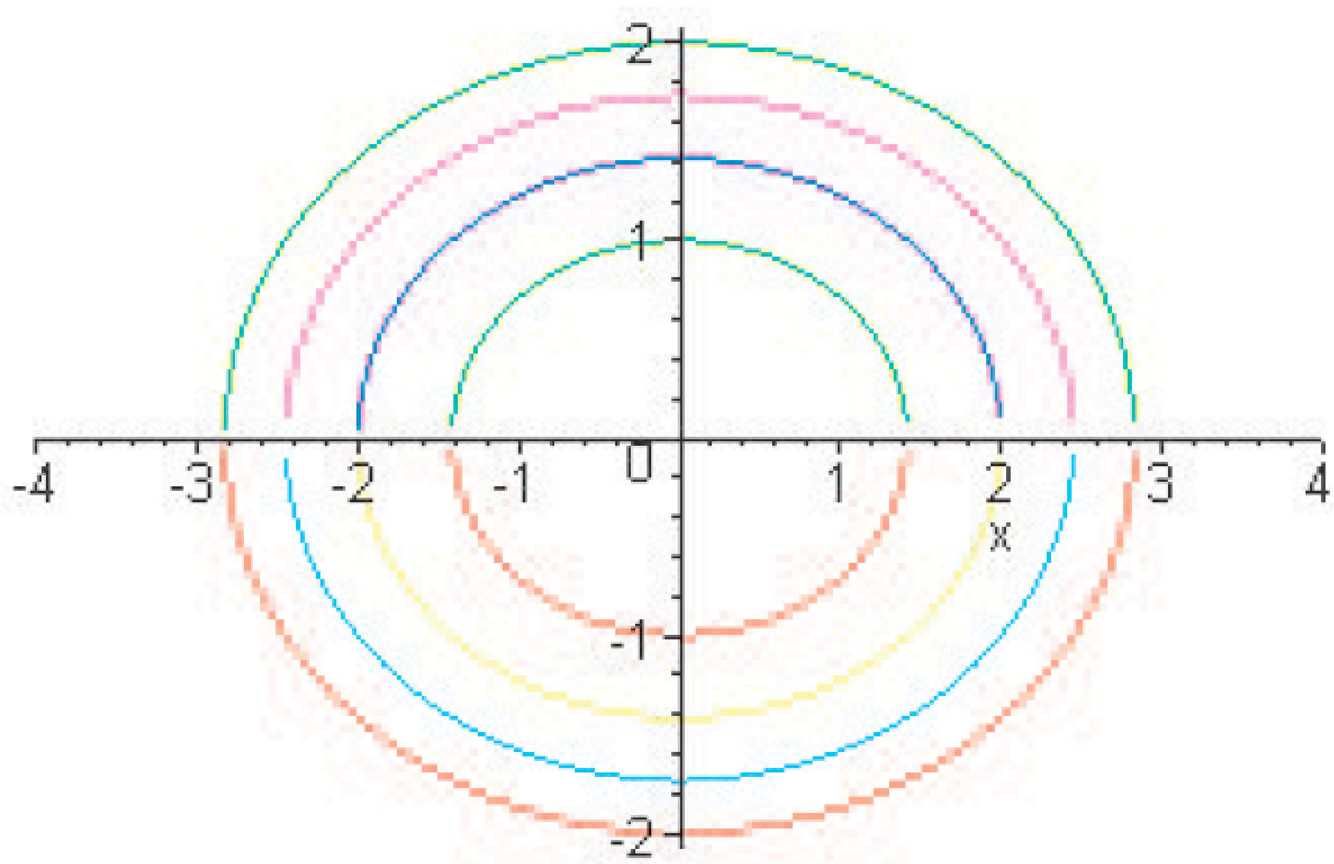
- The solution is

$$2y^2 + x^2 = C_1.$$


2
1
-1
-2
-4
-3
-2
-1
0
1
2
3
4
x

Torricelli's equation

- We found that the equation is of the form

$$y' = k\sqrt{y},$$

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- The general form of the solution is

$$y = \left(\frac{1}{2}kx + C_1\right)^2.$$