3.5: Issues in Curve Sketching

Mathematics 3
Lecture 20
Dartmouth College

February 17, 2010

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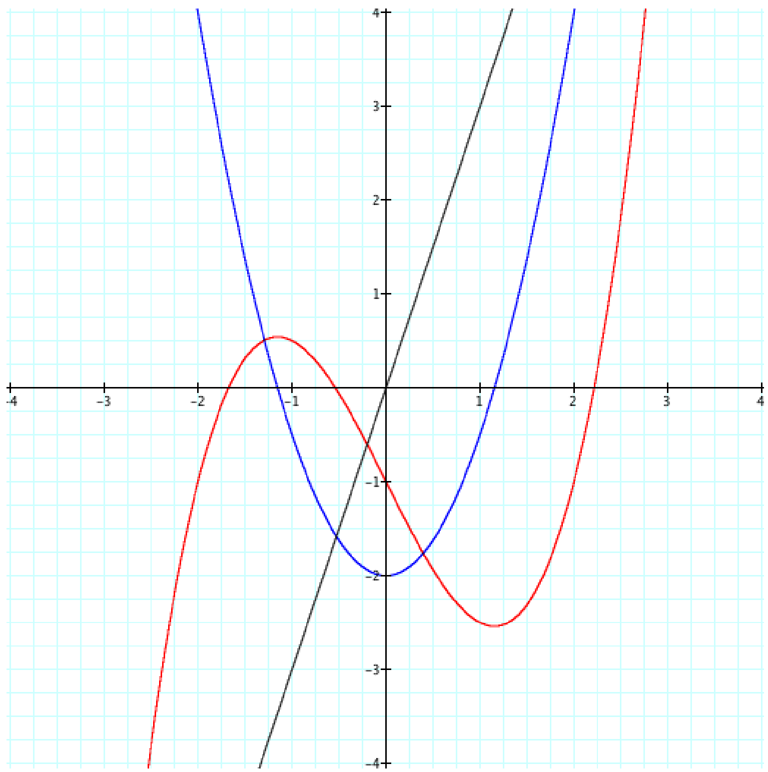
Example 1

Which of the following are the graphs of a function, its derivative and its second derivative?



Answer: $y = \frac{1}{2}x^3 - 2x - 1 \Rightarrow y' = \frac{3}{2}x^2 - 2 \Rightarrow y'' = 3x$

Recall: Monotonicity of Functions on Intervals

Suppose that the function f is defined on an interval I, and let x_1 and x_2 denote points in I:

1. f is increasing on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

2. f is decreasing on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

3. f is **constant** on I if $f(x_1) = f(x_2)$ for *any* $x_1, x_2 \in I$.

Review: Testing Monotonicity via Derivatives

Recall: The derivative function $f'(x)$ tells us the the **slope** of the tangent line to the graph of the function f at the point $(x, f(x))$.

Theorem. (**Increasing/Decreasing Test**) *Let $I = (a, b)$ be an open interval. Suppose that f is differentiable on all of I. Then*

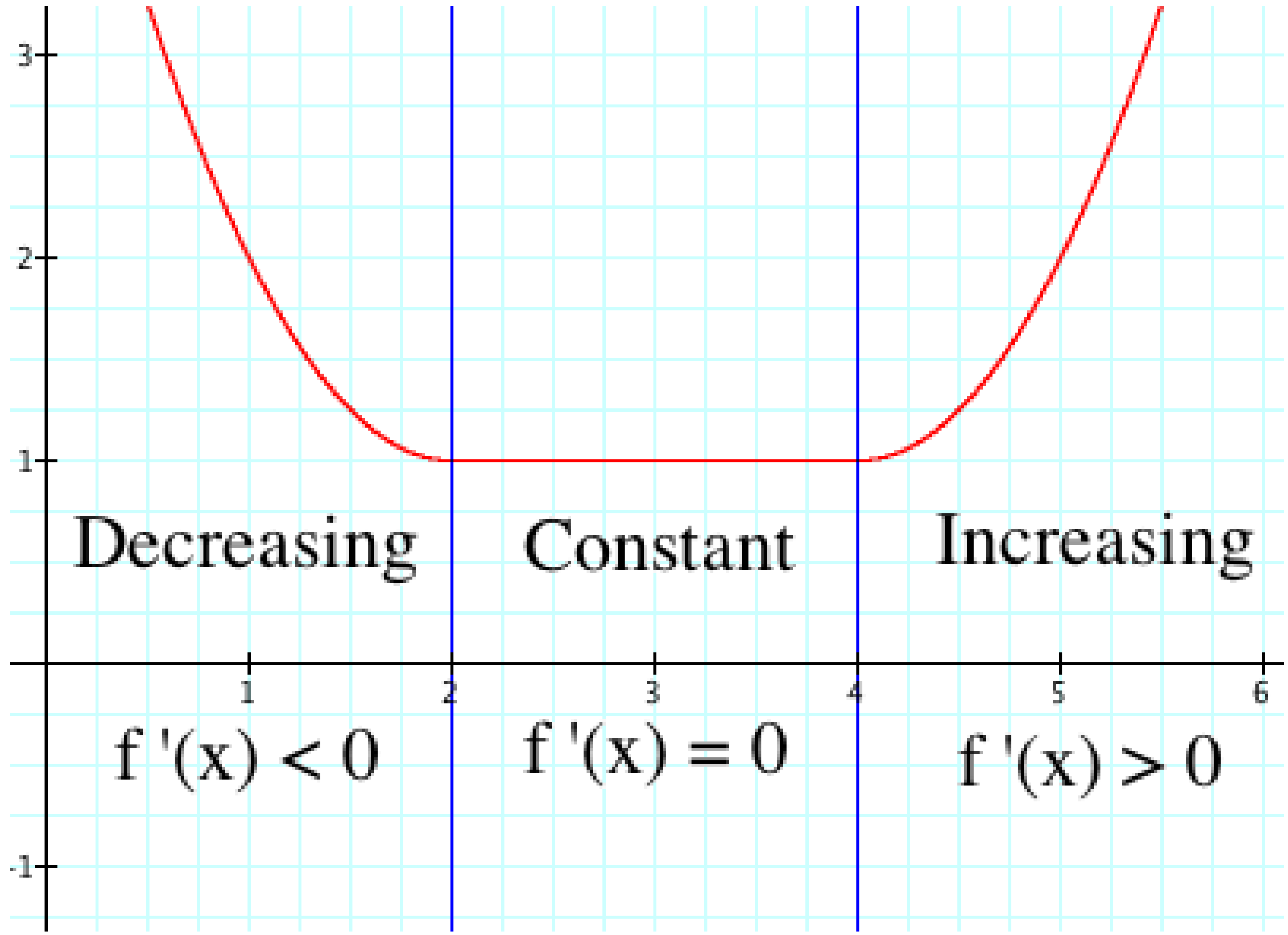
1. If $f'(x) > 0$ for every $x \in I$, then f is increasing on I.

2. If $f'(x) < 0$ for every $x \in I$, then f is decreasing on I.

3. If $f'(x) = 0$ for every $x \in I$, then f is **constant** on I

Review: Testing Monotonicity via Derivatives

Here is how to remember the three cases geometrically:



Recall: The Extreme Value Theorem

Theorem. *If f is continuous on a closed interval $[a, b]$, then there is a point c_1 in the interval where f assumes its maximum value, i.e. $f(x) \leq f(c_1)$ for every x in $[a, b]$, and a point c_2 where f assumes its minimum value, i.e. $f(x) \geq f(c_2)$ for every x in $[a, b]$.*

Zen: A continuous function on a closed and bounded interval $[a, b]$ always has **extreme values** (i.e., max and min) somewhere in the interval. This is an "existence theorem" and is very hard to prove, in generality (Math 35/54/63).

Important Question: How do we FIND these extreme values?

Finding Extreme Values with Derivatives

Theorem. *If f is defined in an open interval (a, b) and achieves a maximum (or minimum) value at a point $c \in (a, b)$ where $f'(c)$ exists, then $f'(c) = 0$.*

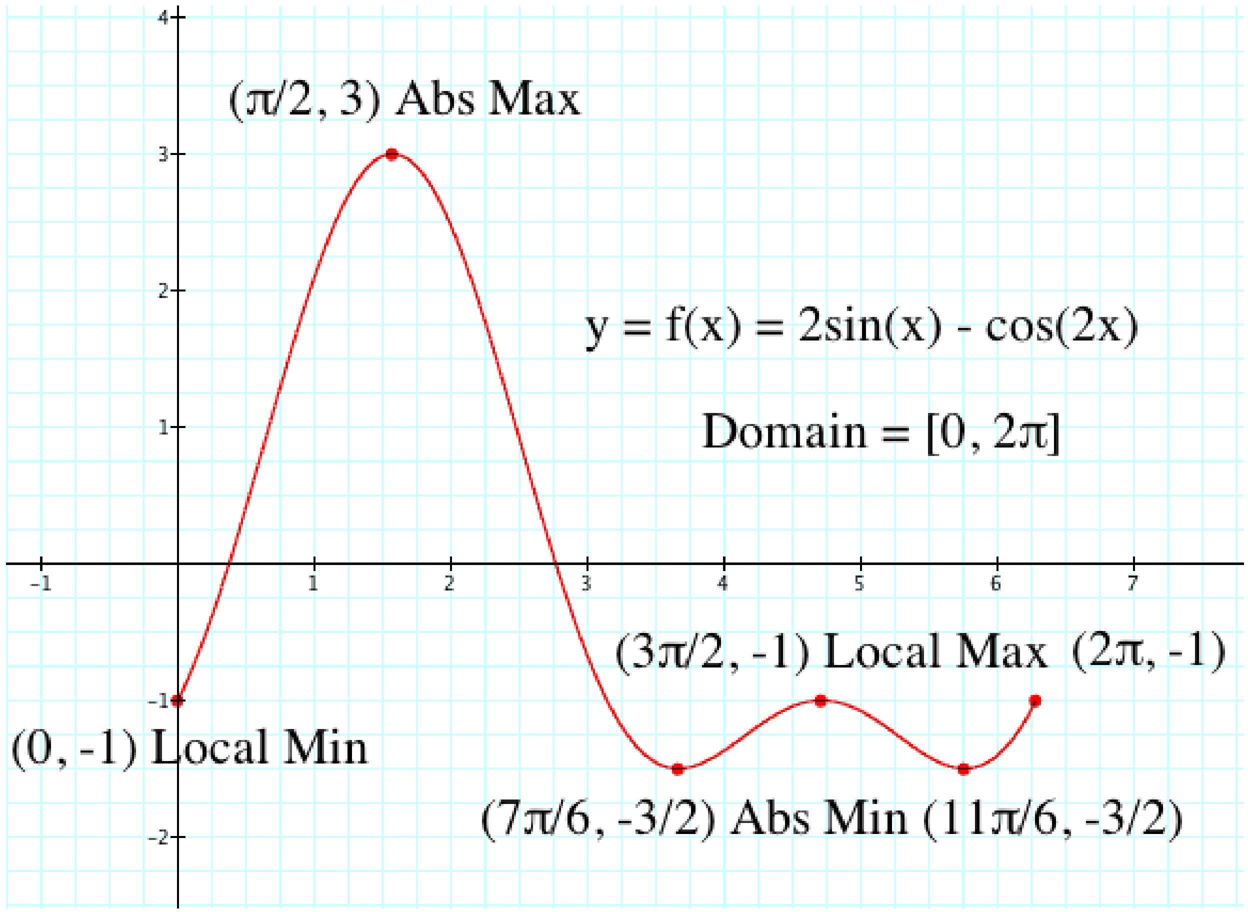
Zen: An **extreme value** (max/min) of a differentiable function in an open interval (a, b) must occur where the graph has a horizontal tangent line. But, just because $f'(c) = 0$ does NOT mean you have an extreme value at $x = c$.

Def: A point $x = c$ in the domain of f where $f'(c) = 0$, or *does not exist*, is called a **critical point** of the function f.

Note: Our textbook calls a point x in the domain of f where $f'(x)$ *does not exist* a **singular point** of f, but most calculus textbooks do not use this!

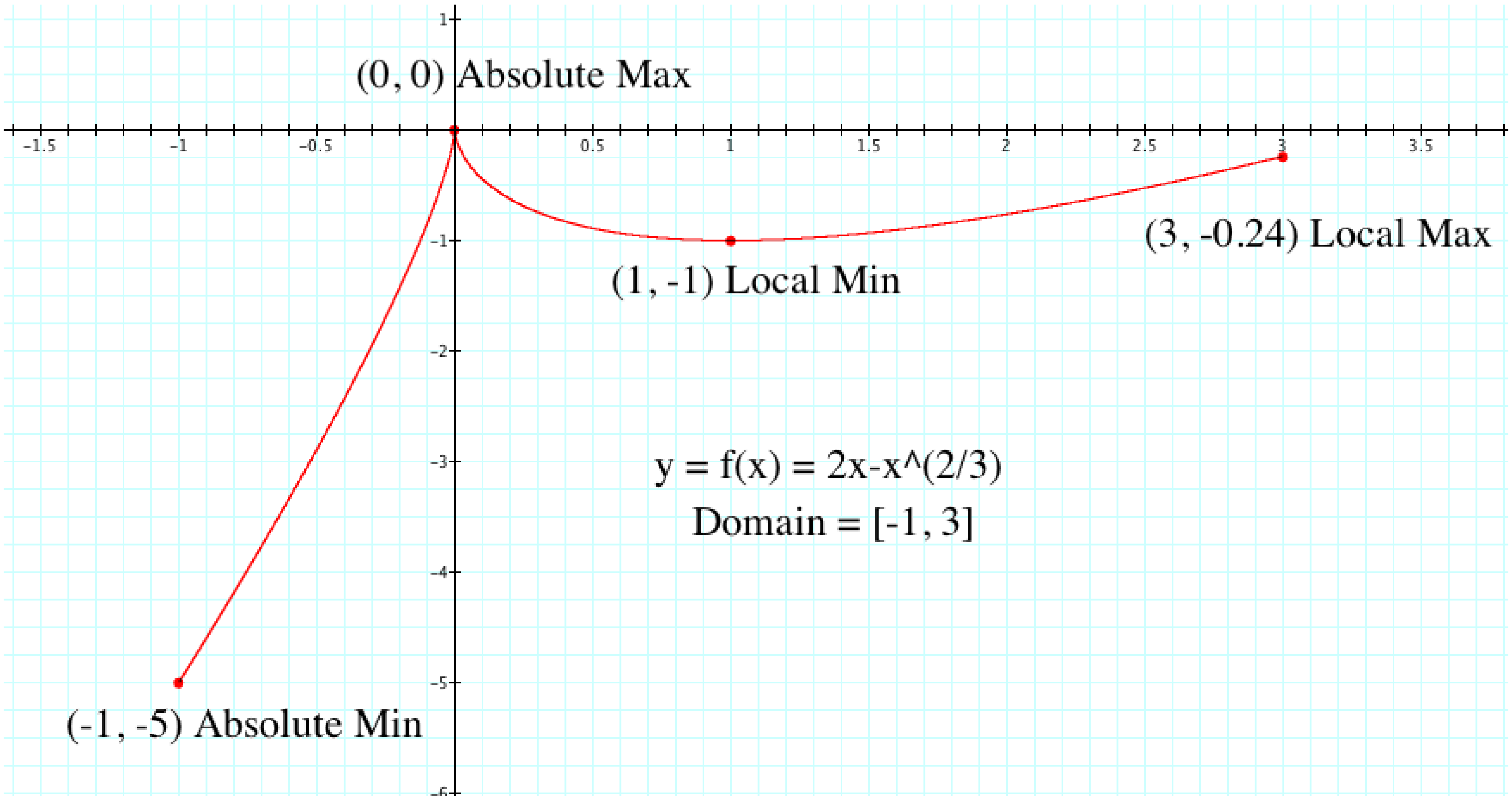
Absolute and Local (Relative) Extrema

Extrema come in 2 flavors: Absolute and Local (Relative)



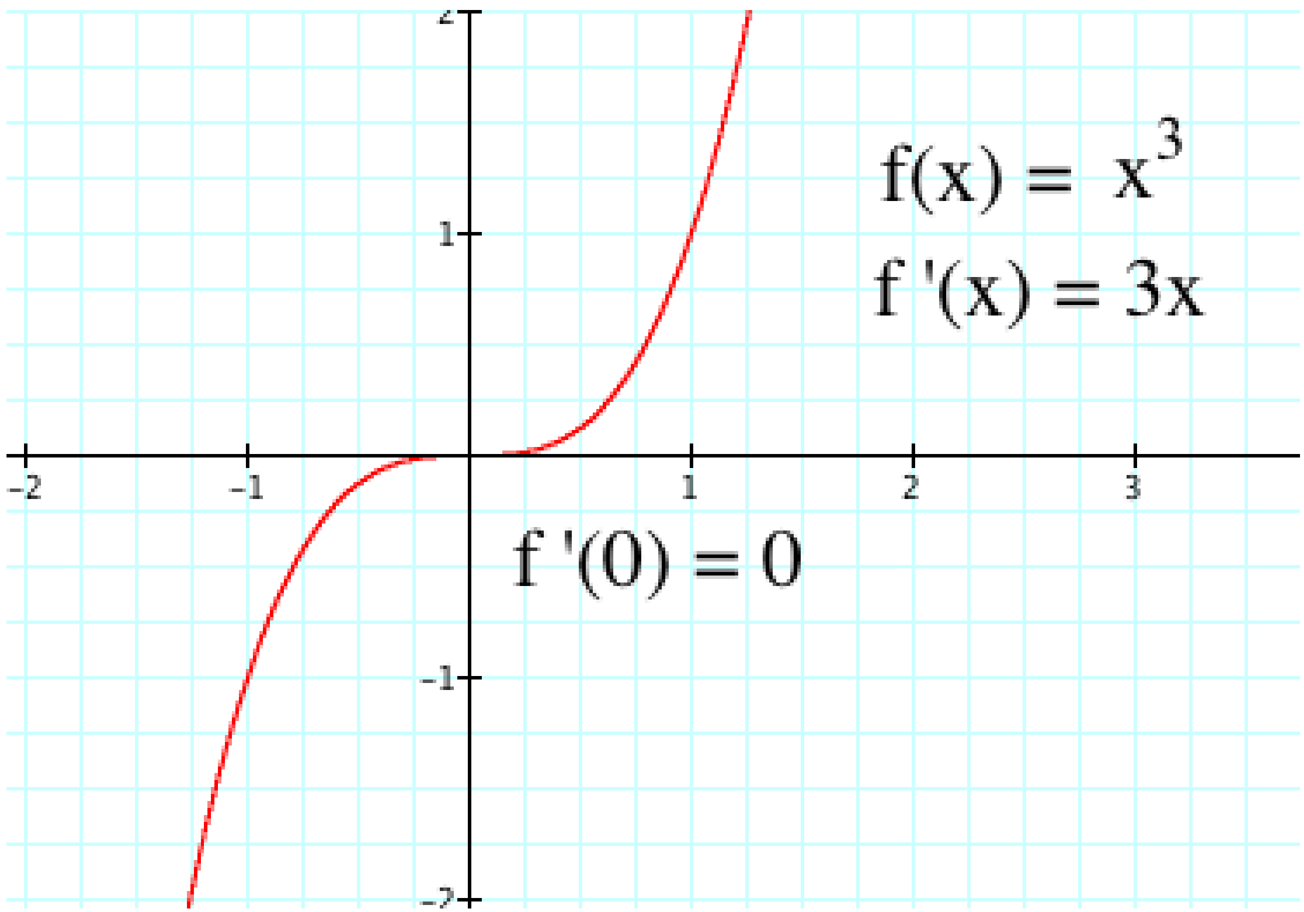
Absolute and Local (Relative) Extrema

We have to check endpoints and critical/singular Points.



Absolute and Local (Relative) Extrema

But just because $f'(x) = 0$ (or DNE) does NOT mean you have a local/abs extremum!



How to find Absolute Extrema on Closed Intervals

To find the (absolute) max and min values of a continuous function $y = f(x)$ on a closed and bounded interval $[a, b]$:

a.) Find the critical/singular points of f inside (a, b).

b.) Evaluate f at each critical/singular point in (a, b).

c.) Evaluate f at the endpoints, i.e., find $f(a)$ and $f(b)$.

d.) The least of these numbers is the **absolute** minimum and the greatest is the **absolute** maximum.

Example 2

Find the extrema of

$$f(x) = 2x - 3x^{2/3}$$

on the interval $[-1, 3]$.

The First Derivative Test (p. 272)

Question: How do we find local (relative) extrema?

Let c be a critical/singular point of a function $y = f(x)$ that is continuous on an open interval $I = (a, b)$ containing c. If f is differentiable on the interval (except possibly at the singular point $x = c$) then the value $f(c)$ can be classified as follows:

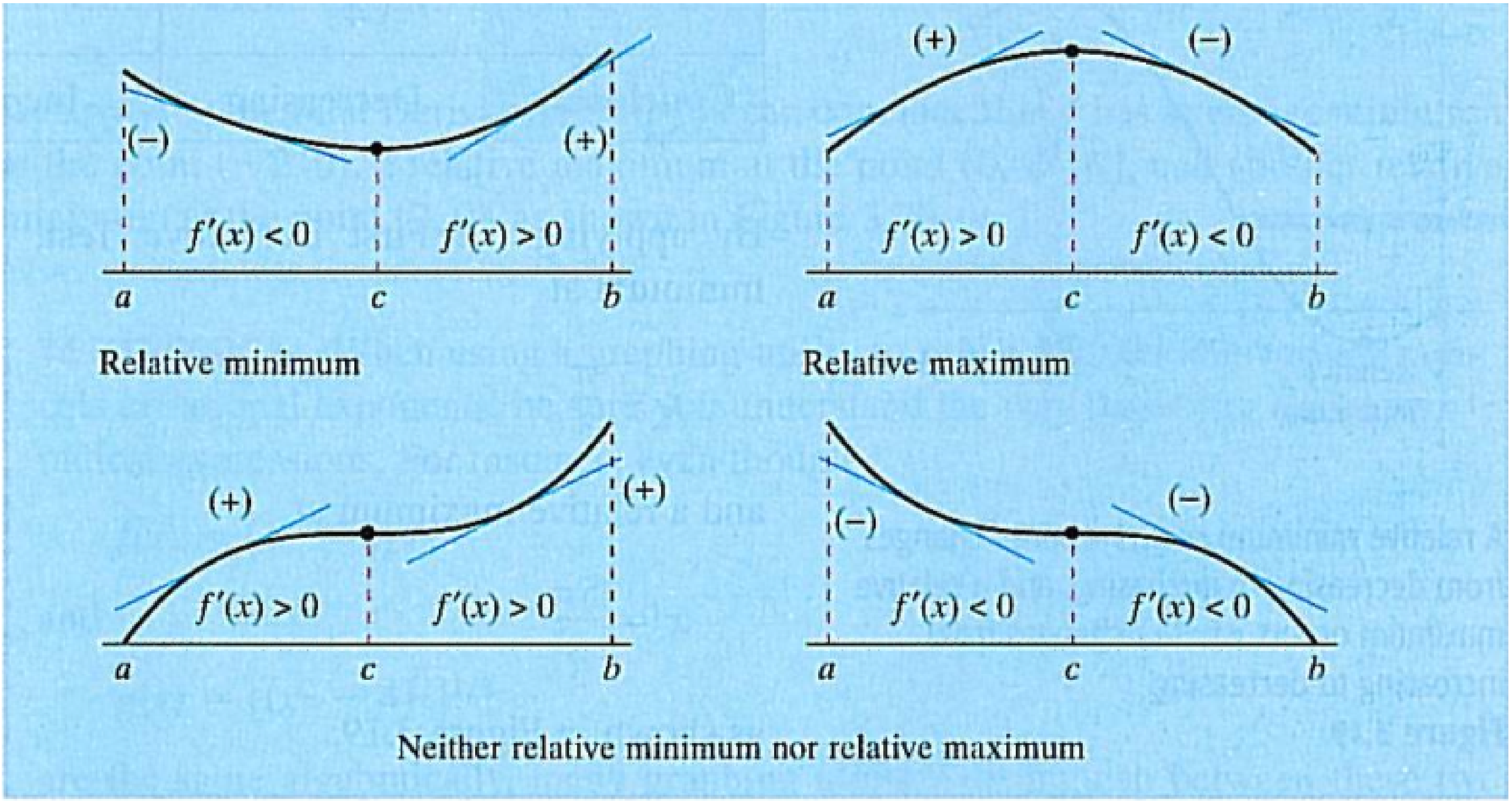
1. If $f'(x)$ changes sign from negative to positive at $x = c$, then $f(c)$ is a **local (relative) minimum**.

2. If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a **local (relative) maximum**.

3. If $f'(x)$ does not have opposite signs on either side of $x = c$, then $f(c)$ is **neither** a local max or min.

The First Derivative Test

Here is a picture that helps to remember the First Derivative Test:



Example 3

Find and classify the local (relative) extrema of the function

$$f(x) = (x-4)x^{\frac{1}{3}}$$

on the whole real line $(-\infty, \infty)$.

Example 4

Find and classify the local (relative) extrema of the function

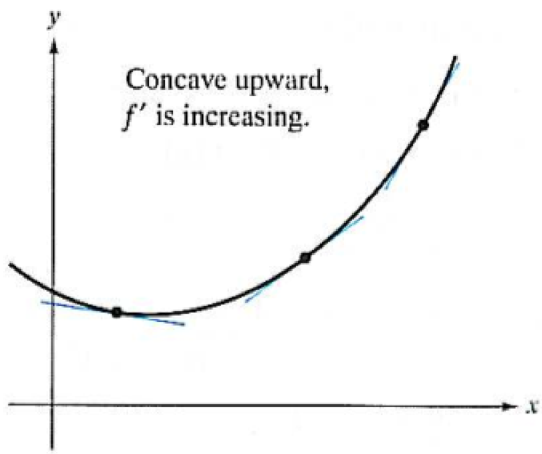
$$f(x) = \frac{x^4 + 1}{x^2}$$

on its natural domain.

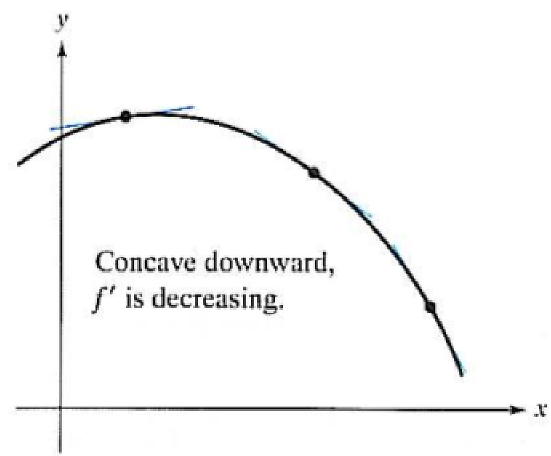
Concavity and Inflection Points

Question: How does the sign of the **second derivative** $f''(x)$ affect the shape of the graph of f?

Def: Let f be a differentiable function on an open interval I. The graph of f is concave upward on I if $f'(x)$ is increasing on I and concave downward on I if $f'(x)$ is decreasing on I.



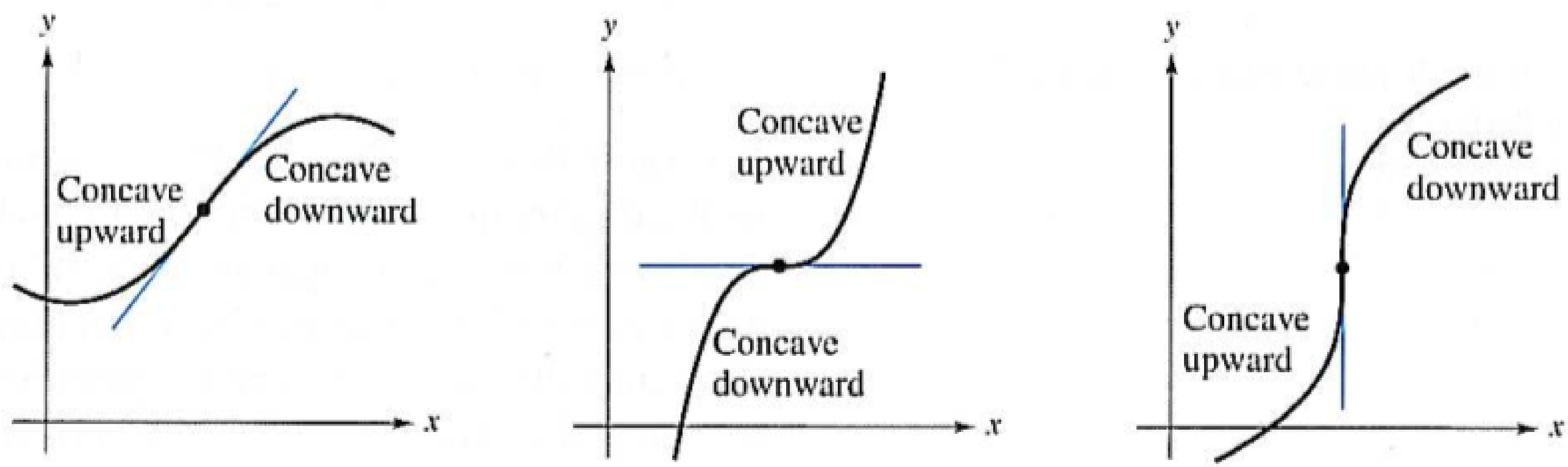
(a) The graph of f lies above its tangent lines.



(b) The graph of f lies below its tangent lines.

Concavity and Inflection Points

Def: The function f has an **inflection point** at the point $x = c$ if it has a tangent line at $x = c$ (e.g., $f'(c)$ exists) and the concavity *changes* at $x = c$ from up to down or vice versa.



The concavity of f changes at a point of inflection.

The Second Derivative Test for Concavity

Since the second derivative $f''(x)$ is the first derivative of $f'(x)$ we easily get:

Theorem 2 (p. 274) Let f be a function whose second derivative f'' exists on an open interval I.

1. If $f''(x) > 0$ on I, then f is concave upward on I.

2. If $f''(x) < 0$ on I, then f is concave downward on I.

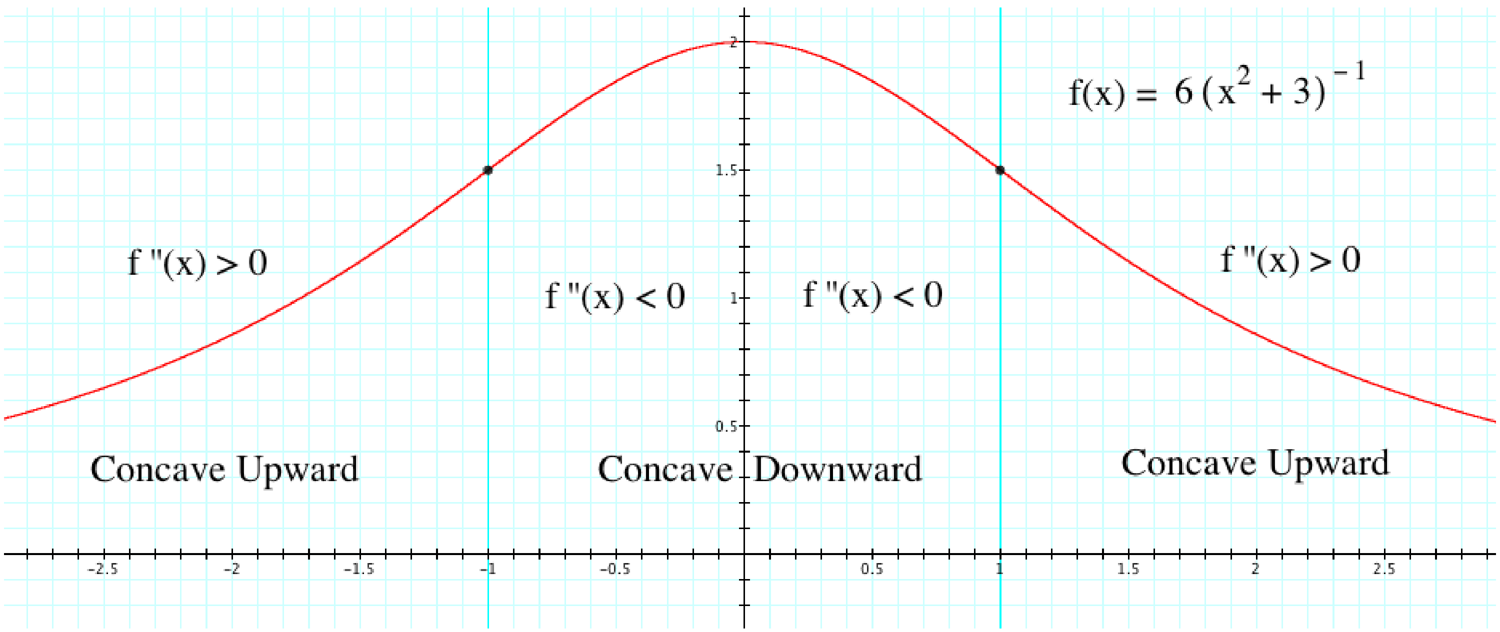
3. If f has an inflection point at x_0 in I and $f''(x_0)$ exists then $f''(x_0) = 0$.

Example 5

Determine the the open intervals on which the function

$$f(x) = 6(x^2 + 3)^{-1}$$

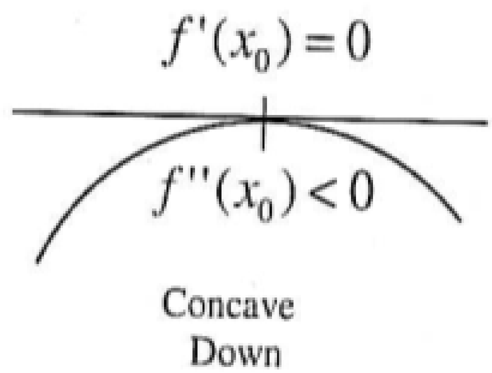
is concave upward or downward and find the inflection points.

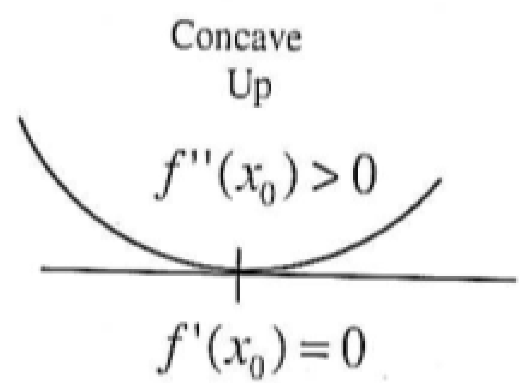


The Second Derivative Test for Local Extrema

Theorem 3 (p. 274) Let f be a function such that the second derivative f'' exists on an open interval I containing x_0.

1. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.
2. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is local maximum.
3. If $f'(x_0) = 0$ and $f''(x_0) = 0$ the test **fails**. Use the First Derivative Test to decide...





Example 6

Find and classify the local extrema of the following functions

a.) $f(x) = x^3 - 12x - 5.$

b.) $h(x) = -3x^5 + 5x^3$