The Fundamental Theorem of Calculus

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Fundamental Theorem of Calculus

Part I-antiderivative: Suppose that f is a continuous function on the interval I containing the point a. Define the function F on I by the integral formula

$$F(x) = \int_a^x f(t)dt.$$

Then F is differentiable on I and $F'(x) = f(x)$. That is, F is an antiderivative of f on I.

Fundamental Theorem of Calculus

Part II-evaluation: If $G(X)$ is any antiderivative of f on I (that is, $G'(x) = f(x)$ on I), then for any b in I,

$$\int_a^b f(x)dx = G(b) - G(a).$$

Examples

- $\displaystyle\int_0^1 (x+1)dx$
- $\displaystyle\int_0^{\pi/4} \sin x dx$
- $\displaystyle\int_0^{\pi/4} \sec^2 x$

- $\dfrac{d}{dx}\displaystyle\int_1^x t^2 dt$

- $\dfrac{d}{dx}\displaystyle\int_1^{x^2} t^3 dt$

- $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} e^{-t^2} dt$

Techniques of Integration

$$
\begin{aligned}
\int u^r \, du &= \frac{u^{r+1}}{r+1} + C, r \neq -1 \\
\int \frac{1}{u} \, du &= \ln |u| + C \\
\int \sin u \, du &= -\cos u + C \\
\int \cos u \, du &= \sin u + C \\
\int \sec^2 u \, du &= \tan u + C \\
\int \sec u \tan u \, du &= \sec u + C \\
\int e^u \, du &= e^u + C
\end{aligned}
$$

The Method of Substitution

- Differentials continue to be a very useful technique for solving integrals.

- If $y = x^3$, then $dy = 3x^2 dx$.

- If $y = \sin 4x$, then $dy = 4 \cos 4x dx$.

Reversing the Chain Rule

- If $u = g(x)$ is a function of x, and f is a function of u, then the chain rule tells us that

$$(f(g(x)))' = f'(g(x))g'(x).$$

- Integrating the right hand side reverses the chain rule and we get

$$\int f'(g(x))g'(x)dx = f(g(x)) + C.$$

- Substitute $u = g(x)$ and the differential $du = g'(x)dx$. When we make these two substitutions we get

$$\int f'(u)du = f(u) + C.$$

Examples

- $\int e^{7x} dx$
- $\int \sin 2x dx$
- $\int \frac{x}{x^2+1} dx$
- $\int \frac{x^2+1}{x} dx$
- $\int \frac{x^2+1}{x^3+3x+2} dx$

- $\displaystyle\int \frac{\ln x}{x} dx$

- $\displaystyle\int_e^{e^2} \frac{\ln x}{x} dx$

- $\displaystyle\int_0^{\pi/4} \tan x dx$

Integration by Parts

- Another technique of integration that is often useful involves an undoing of the product rule.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int udv = uv - \int vdu.$$

Example

- $\int xe^x dx$
- $\int \ln x dx$
- $\int_1^e \ln x dx$
- $\int x^2 \sin x dx$
- $\int e^x \sin x dx$