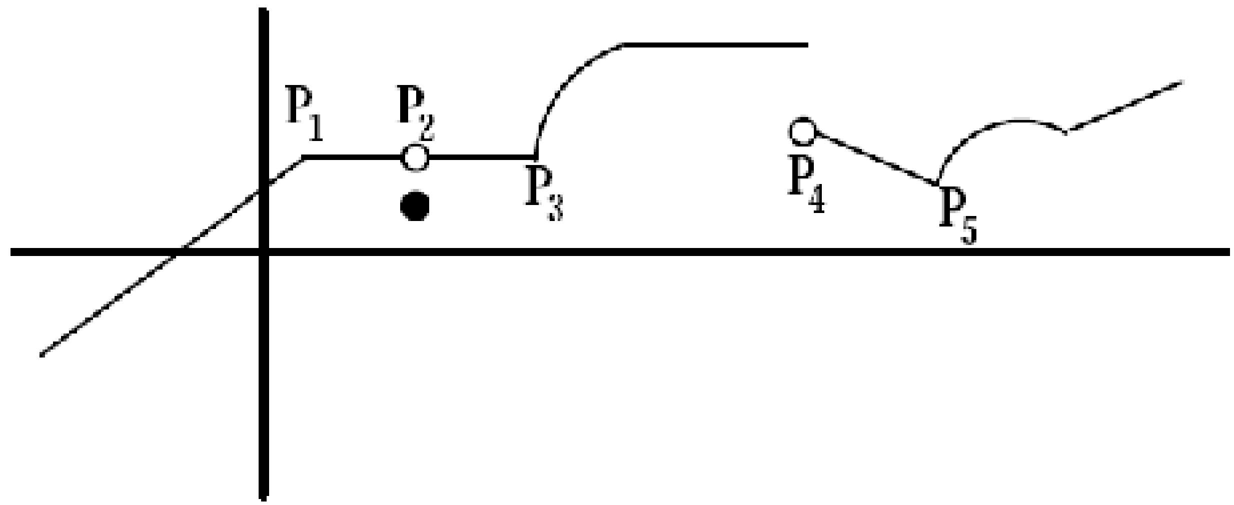
Continuity

10/07/2005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Interior Point

An *interior point* of a set of real numbers is a point that can be enclosed in an open interval that is contained in the set.

Definition

- A function is continuous at an interior point c of its domain if $\lim_{x \to c} f(x) = f(c)$.

- If it is not continuous there, i.e. if either the limit does not exist or is not equal to $f(c)$ we will say that the function is discontinuous at c.

Note:

1. The function f is defined at the point $x = c$,

2. The point $x = c$ is an interior point of the domain of f,

3. $\lim_{x \to c} f(x)$exists, call it L, and

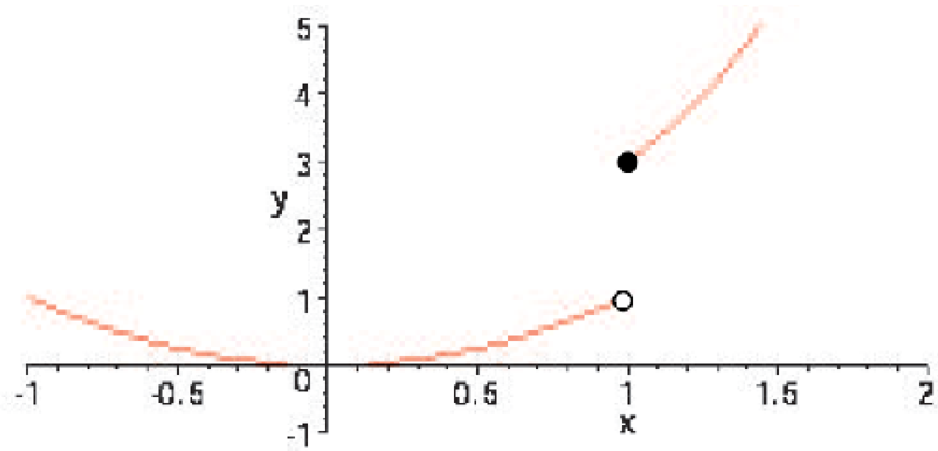
4. $L = f(c)$.

Example

Is the function

$$f(x) = \begin{cases} x^2 & x < 1 \\ x^3 + 2 & 1 \leq x \end{cases}$$

continuous at $x = 1$?



Right Continuity and Left Continuity

- A function f is right continuous at a point c if it is defined on an interval $[c, d]$ lying to the right of c and if $\lim_{x \to c^+} f(x) = f(c)$.

- Similarly it is left continuous at c if it is defined on an interval $[d, c]$ lying to the left of c and if $\lim_{x \to c^-} f(x) = f(c)$.

Definition

A function f is continuous at a point $x = c$ if c is in the domain of f and:

1. If $x = c$ is an interior point of the domain of f, then $\lim_{x \to c} f(x) = f(c)$.

2. If $x = c$ is not an interior point of the domain but is an endpoint of the domain, then f must be right or left continuous at $x = c$, as appropriate.

- A function f is said to be a continuous function if it is continuous at every point of its domain.

- A point of discontinuity of a function f is a point in the domain of f at which the function is not continuous.

Facts

- All polynomials,
- Rational functions,
- Trigonometric functions,
- The absolute value function, and
- The exponential and logarithm functions

are continuous.

Example

- The rational function $f(x) = \frac{x^2-4}{x-2}$ is a continuous function.
- The domain is all real numbers except 2.
- $\lim_{x\to 2} f(x) = 4$ exists.

It has a *continuous extension*

$$F(x) = \begin{cases} f(x) & \textit{if } x \textit{ is in the domain of } f \\ 4 & \textit{if } x = 2. \end{cases}$$

Example

The function

$$f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$$

is discontinuous at $\pi/3$.

We can “remove” the discontinuity by redefining the value of f at $\pi/3$.

Definition

- If c is a discontinuity of a function f, and if $\lim_{x \to c} f(x) = L$ exists, then c is called a removable discontinuity. The discontinuity is removed by defining $f(c) = L$.

- If f is not defined at c but $\lim_{x \to c} f(x) = L$ exists, then f has a continuous extension to $x = c$ by defining $f(c) = L$.

Example

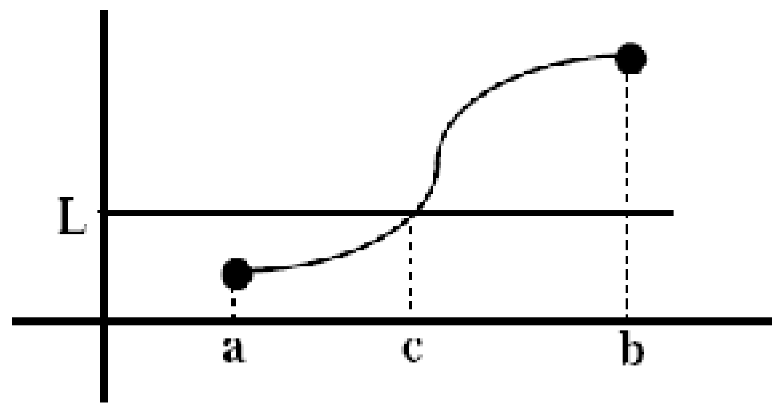
Suppose that $f(x)$ is defined piecewise as

$$f(x) = \begin{cases} -x^2 + 1 & x < 2 \\ x + k & x > 2 \end{cases}$$

Let us find a value of the constant k such that f has a continuous extension to $x = 2$.

The Intermediate Value Theorem

If a function f is continuous on a closed interval $[a, b]$, and if $f(a) < L < f(b)$ (or $f(a) > L > f(b)$), then there exists a point c in the interval $[a, b]$ such that $f(c) = L$.



Example

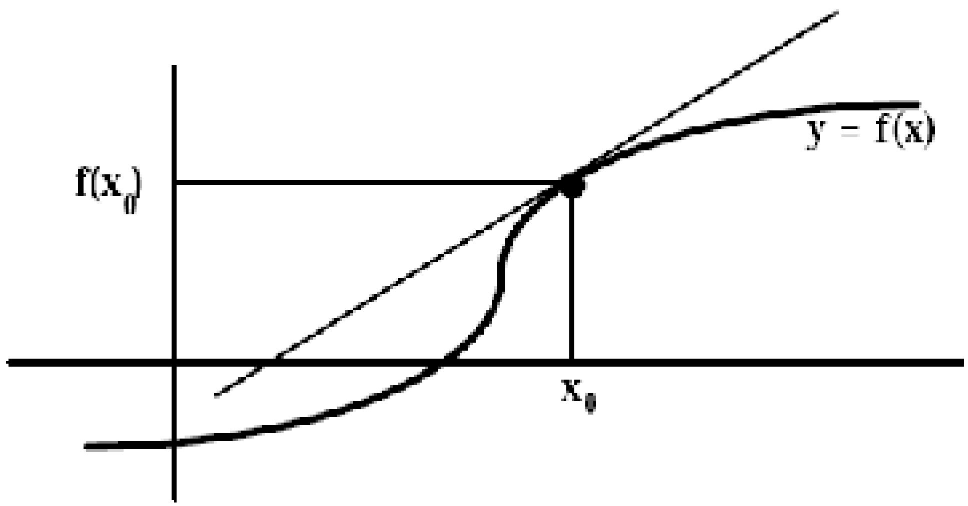
Show that the equation $x^5 - 3x + 1 = 0$ has a solution in the interval $[0, 1]$.

Example

Does the equation $1/x = 0$ have a solution?

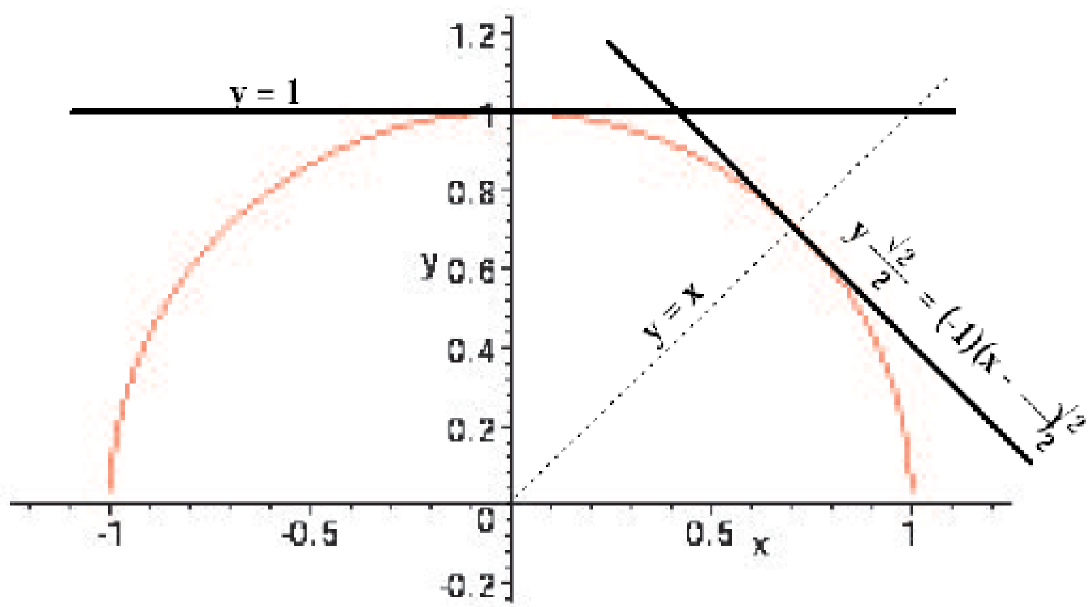
The Tangent Line and Their Slope

- **The Tangent Line Problem** Given a function $y = f(x)$ defined in an open interval and a point x_0 in the interval, define the tangent line at the point $(x_0, f(x_0))$ on the graph of f.



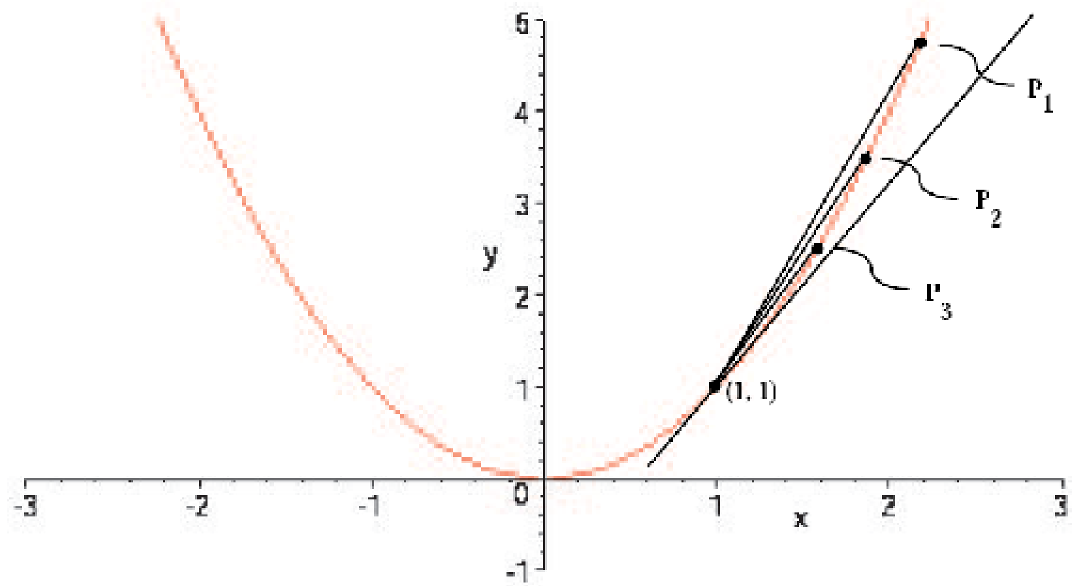
Example

Find the equations of the tangent lines to the graph of $f(x) = \sqrt{1-x^2}$ at the points $(0, 1)$ and $(\frac{\sqrt{2}}{2}.\frac{\sqrt{2}}{2})$.



Example

Let $f(x) = x^2$.



Definition

Given a function f and a point x_0 in its domain, the slope of the tangent line at the point $(x_0, f(x_0))$ on the graph of f is

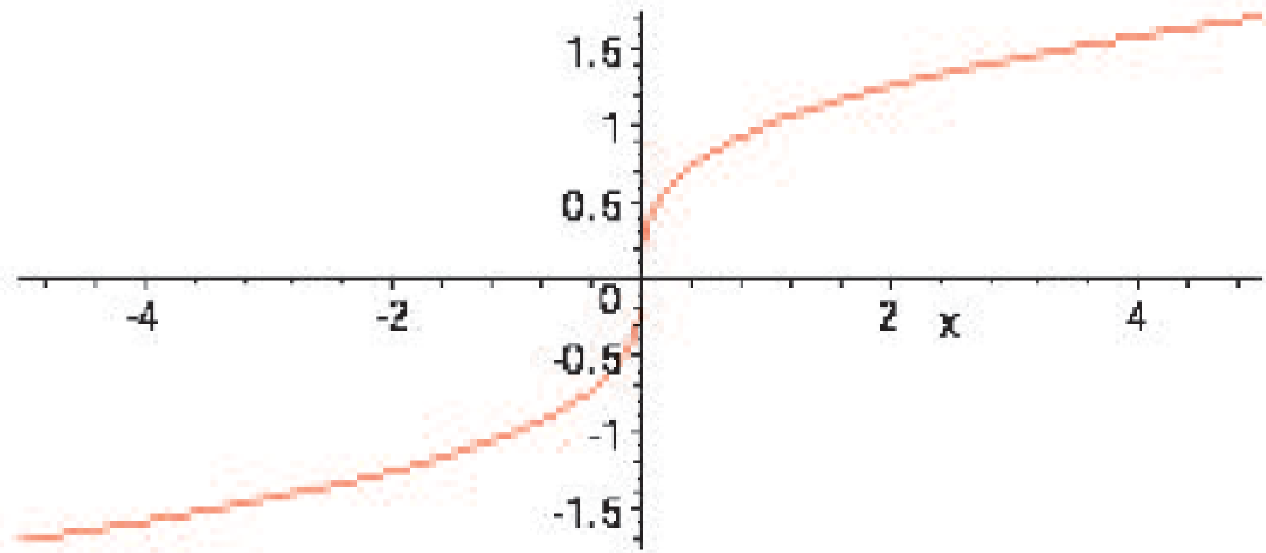
$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example

Given $f(x) = \sqrt{x}$, find the equation of the tangent line at $x = 4$.

Example

Find the tangent line to the graph of $f(x) = x^{1/3}$ at $x = 0$.



Example

Let f be the piecewise defined function

$$f(x) = \begin{cases} 2 - x^2 & x \leq 1 \\ x^3 & x > 1 \end{cases}$$

Is the function continuous, and does it have a tangent line at $x = 1$?