2.11: Implicit Differentiation (cont'd) and 2.12: Derivatives of Exponential and Logarithm Functions

Mathematics 3
Lecture 12
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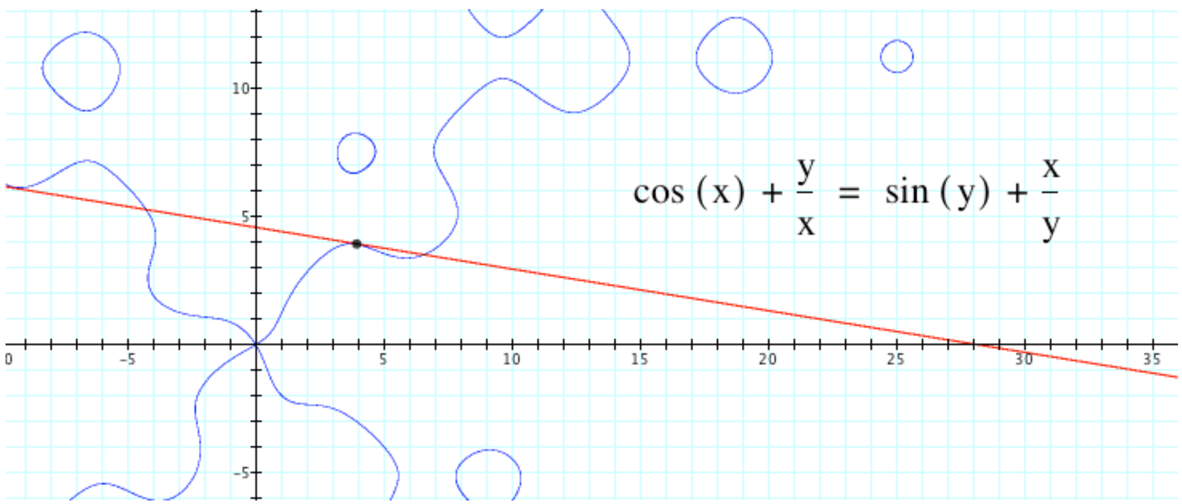
Implicit Differentiation (cont'd)

Given a curve $F(x, y) = k$ (where k is a constant) which implicitly defines an (unknown..?) function $y = f(x)$, we can find the tangent slopes $\frac{dy}{dx}$ at points (x, y) on the curve via implicit differentiation.



Example 10 (yesterday): $y' = \frac{dy}{dx} = \frac{y^3+x^2y+x^2y^2\sin(x)}{x^3+xy^2-x^2y^2\cos(y)}$

Implicit Differentiation (cont'd)

For example, with the curve $x^3y^3 = 1$, we would proceed as follows:

1. Differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}\left(x^3y^3\right) = \frac{d}{dx}(1)$$

2. Use derivative rules on x and y stuff, but also use Chain Rule on y stuff:

$$\begin{aligned} \frac{d}{dx}\left(x^3\right)y^3 + x^3\frac{d}{dx}\left(y^3\right) &= 0 \\ 3x^2y^3 + x^3(3y^2y') &= 0 \end{aligned}$$

3. Solve for y' in terms of x and y:

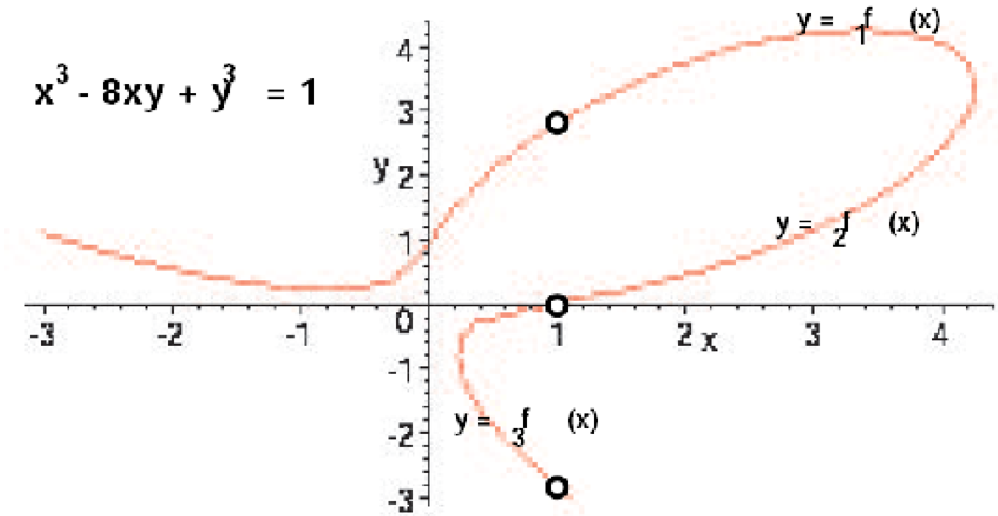
$$\frac{dy}{dx} = y' = \frac{-3x^2y^3}{3x^3y^2} = -\frac{y}{x}$$

Example 0 (Ex 11 from yesterday)

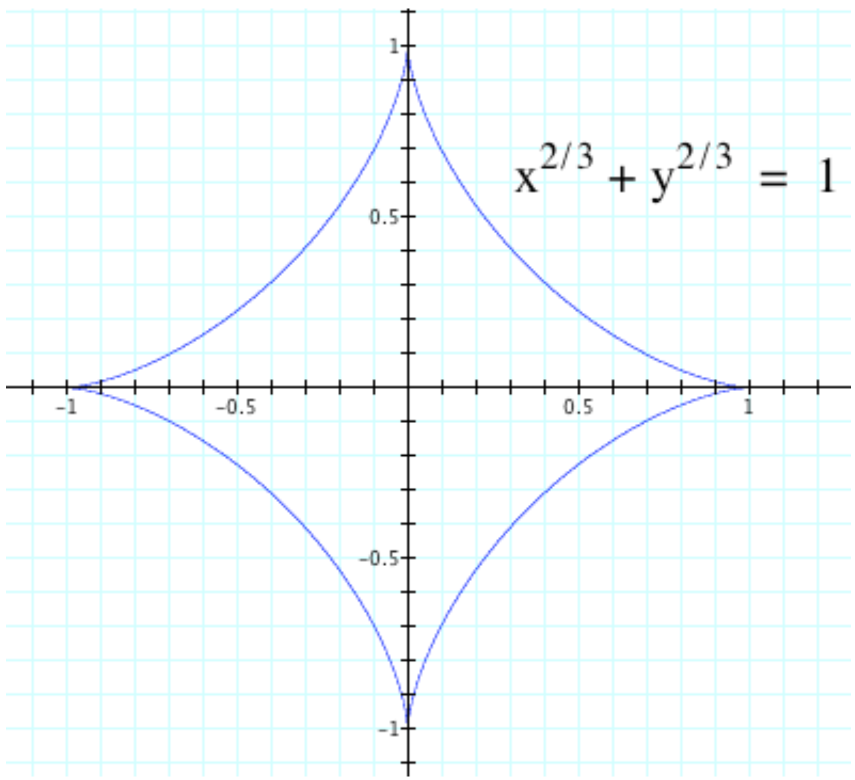
We mentioned yesterday the equation

$$x^3 - 8xy + y^3 = 1.$$

Find the slope at the points on the curve for which $x = 1$.



The Astroid Curve



This curve is related to the orbit of a moon around a planet...

The Astroid Curve (cont'd)

Let us now find the slopes of tangent lines at points (x, y) on the astroid curve

$$x^{2/3} + y^{2/3} = 1$$

$$\begin{aligned}
\frac{d}{dx}\left(x^{2/3} + y^{2/3}\right) &= \frac{d}{dx}(1) \\
\frac{2}{3}x^{-1/3} + \frac{d}{dx}\left(y^{2/3}\right) &= 0 \\
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} &= 0 \\
\text{Solve for } \frac{dy}{dx} \Longrightarrow \frac{dy}{dx} &= -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}
\end{aligned}$$

Derivatives and Inverse Functions

Suppose a differentiable function f has an inverse f^{-1}. Find the derivative of f^{-1} in terms of the derivative of f.

$$\begin{aligned} y = f^{-1}(x) \quad &\Longleftrightarrow \quad x = f(y) \\ \frac{d}{dx}\Big(x\Big) \quad &= \quad \frac{d}{dx}\Big(f(y)\Big) \\ 1 \quad &= \quad f'(y)\frac{dy}{dx} \\ \frac{dy}{dx} \quad &= \quad \frac{1}{f'(y)} \\ \Big[f^{-1}(x)\Big]' \quad &= \quad \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

Example 1 Show that $f(x) = x^3 + x - 7$ has an inverse function and, noting that $f(2) = 3$, find $(f^{-1})'(3)$.

The Derivative of The Exponential Function

Recall: In section 2.1, we studied the limit

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

We will use this to compute the derivative of $y = e^x$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\
&= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

Basic Formula:

$$\frac{d}{dx}(e^x) = e^x$$

Theorem. *Let u be a (possibly unknown) function of x. Then*

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

Example 2

Find the following:

a.) If $y = e^{-5x^2+4}$, then find y''.

b.) Let $f(t) = e^{\sin t}$. Find $f'(\frac{\pi}{4})$.

c.) Find dx/dy for the curve $e^x \tan(xy^2) = x + 3y$.

d.) Find the line tangent to the curve

$$(x + y)^2 = ye^x$$

at the point $(0, 1)$.

The Derivative of the Natural Logarithm

We can find the derivative of $y = \ln x$ by implicit differentiation:

$$\begin{aligned}
y = \ln x \quad &\Leftrightarrow \quad e^y = x \\
\frac{d}{dx}(e^y) \quad &= \quad \frac{d}{dx}(x) \\
e^y \frac{dy}{dx} \quad &= \quad 1 \\
\frac{dy}{dx} \quad &= \quad \frac{1}{x} \\
\frac{d}{dx}\big(\ln x\big) \quad &= \quad \frac{1}{x}
\end{aligned}$$

The Derivative of the Natural Logarithm

Theorem. *Let u be a (possibly unknown...) function of x. Then*

$$\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}.$$

Example 3 Find the following:

a.) Given $y = \ln x^2$, find y' in 2 different ways.

b.) Let $G(t) = (1 + \ln|\sin t|)^7$. Find $G'(t)$.

The Calculus Standards: e^x and $\ln x$

The natural exponential and natural functions are the "standard" since all other exponential and logarithmic functions, and their derivatives, can be found using them:

$$\begin{aligned} a^x &= e^{x \ln a} \Rightarrow \frac{d}{dx}(a^x) = a^x(\ln a) \\ \log_a x &= \frac{\ln x}{\ln a} \Rightarrow \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \end{aligned}$$

Example 3 Find the following:

- If $w = 4^x$ find $D_2 w$.
- Show that $f(x) = x^\pi - \pi^x$ has negative (tangent) slope at $x = \pi$.
- If $y = x^x$ then find dy/dx.

The Equation $y' = ky$

- Suppose y is a function of x and satisfies the equation

$$y' = ky$$

- If $k = 1$, then $y = e^x$ has this property and thus solves the equation.

- In fact $y = e^{kx}$ solves the equation for any k.

- The equation $y' = ky$ is a differential equation and is very important in population models and exponential growth/decay problems that we will see later. (So stay tuned....)

Remember: The midterm exam is Monday!

"First you forget logarithms. Then you forget how to do long division. Then the multiplication table begins to go . . ."