# 2.12: Derivatives of Exp/Log (cont'd) and 2.15: Antiderivatives and Initial Value Problems

Mathematics 3
Lecture 14
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# Derivatives of the Exponential and Logarithmic Functions

**Recall:** We have the following formulas:

$$\begin{aligned}\frac{d}{dx}(e^x) &= e^x \\ \frac{d}{dx}\ln|x| &= \frac{1}{x}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(e^u) &= e^u\,\frac{du}{dx} \\ \frac{d}{dx}\ln|u| &= \frac{1}{u}\,\frac{du}{dx}\end{aligned}$$

# The Calculus Standards: $e^x$ and $\ln x$

The natural exponential and natural functions are the "standard" since all other exponential and logarithmic functions, and their derivatives, can be found using them:

$$\begin{aligned} a^x &= e^{x \ln a} \Rightarrow \frac{d}{dx}(a^x) = a^x(\ln a) \\ \log_a x &= \frac{\ln x}{\ln a} \Rightarrow \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \end{aligned}$$

**Example 1:** Find the following:

- If $w = 4^x$ find $D_2 w$.
- Show that $f(x) = x^\pi - \pi^x$ has negative (tangent) slope at $x = \pi$.
- If $y = x^x$ then find $y'(x)$.

# The Equation $y' = ky$

- Suppose $y$ is a function of $x$ and satisfies the equation

$$y' = ky$$

- If $k = 1$, then $y = e^x$ has this property and thus solves the equation.

- In fact $y = e^{kx}$ solves the equation for any $k$.

- The equation $y' = ky$ is a differential equation and is very important in population models and exponential growth/decay problems that we will see later.

# The Concept of the Antiderivative

An **antiderivative** of a function $f$ on an interval $I$ is another function $F$ such that

$$F'(x) = f(x)$$

for all $x \in I$. That is, the derivative of $F$ is the given function $f$:

$$\frac{dF}{dx} = f(x)$$

The derivative of an antiderivative is the **original** function $f$.

**Example 2:** Consider the function

$$f(x) = x^3 - 2\sin(2x) + e^{5x} + 1.$$

Find two **different** antiderivatives of $f$, say $F(x)$ and $G(x)$.

**Theorem.** *Suppose that $h$ is differentiable in an interval $I$ and*

$$h'(x) = 0$$

*for all $x \in I$. Then $h$ is a constant function; i.e.*

$$h(x) = C$$

*for all $x \in I$, where $C$ is a constant.*

Okay, this is duhhh...obvious, but why do we care?

Suppose $F(x)$ and $G(x)$ are ANY two antiderivatives of $f(x)$ on $I$:

$$\frac{d}{dx}(F(x) - G(x)) = F'(x) - G'(x) = f(x) - f(x) = 0$$

$$\implies F(x) - G(x) = C \quad (Constant)$$

- If $F(x)$ is one antiderivative of $f(x)$, then any *other* antiderivative $G(x)$ must be of the form

$$G(x) = F(x) + C,$$

where $C$ is some *specific* constant (e.g., $C = 7$).

- We refer to $F(x) + C$ (where $C$ is an **arbitrary** constant) as the (most) general antiderivative and denote it by

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

- We call this the indefinite integral of $f$ (with respect to $x$).

  indefinite integral$=$ (most) general form(ula) of an antiderivative

# Basic ("atomic") Indefinite Integrals (p. 201)

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

**NB:** Each integral formula has it's own (dual) derivative formula...

# Indefinite Integrals

We can express this relationship via a *very important formula*:

$$\frac{d}{dx}\int f(x)\,\mathsf{d}x = f(x)$$

But, we also have

$$\int \frac{d}{dx}f(x)\,\mathsf{d}x = \int f'(x)\,\mathsf{d}x = f(x) + C$$

and we may NOT know which constant $C$ we may need to choose...especially if we DO NOT KNOW the original function $f(x)$.

**Example 3:** Find the function $f(x)$ whose derivative is $6x^2 - 1$ for all real $x$, and for which $f(1) = 3$.

# Linearity of Indefinite Integrals

**Theorem.** *Suppose the functions $f$ and $g$ both have antiderivatives on the interval $I$. Then for any constant $a$, the function $af + g$ also has an antiderivative on $I$ and*

$$\int (af + g)dx = a \int f(x)dx + \int g(x)dx$$

**Example 4:** Find the function $g(t)$ whose derivative is

$$\frac{t+5}{t^{3/2}}$$

and whose graph passes through the point $(4, -7)$.

# Example 5

Compute the indefinite integral

$$\int \left(5x^6 + 3\sqrt{x} + e^{x-2} - 3\sec(x)\tan(x)\right) \mathsf{d}x$$

**NOTE:** Finding an antiderivative of $f(x)$ can be thought of as solving the equation $\frac{dy}{dx} = f(x)$ for the unknown function $y$ (which is the antiderivative). We can rephrase the previous problem as:

**Dual Example 5** Find the most general solution to the equation

$$y' = 5x^6 + 3\sqrt{x} + e^{x-2} - 3\sec(x)\tan(x).$$

# (Ordinary) Differential Equations (ODEs)

- These are equations that involve one or more derivatives of an unknown function $y$ are called *differential equations*. The order of the ODE is the order of the highest derivative in the equation.

- Solving a differential equation means finding a function $F(x)$ that satisfies the equation identically when substituted for the unknown function $y$.

- Differential equations are important in physics, biology, chemistry, economics, engineering, etc. Math 23 is a course devoted only to solving differential equations!

# Example 6

Consider the first-order differential equation

$$y' = 6x^2 - 1.$$

a.) Find the (most) **general solution** to the differential equation.

b.) Find the solution that satisfies the initial value $y(1) = 3$.

**NOTE:** Part (b) is just a *rephrasing* of Example 3 above!

# Example 7

Consider the second-order differential equation

$$x^2y'' - xy' - 3y = 0$$

a.) Show that for any constants $A$ and $B$ the function

$$y = Ax^3 + \frac{B}{x}$$

is a solution on any interval not containing $x = 0$.

b.) Find the *particular solution* that satisfies the initial values:

$$\begin{cases} y(1) & = & 2 \\ y'(1) & = & -6. \end{cases}$$

# Initial Value Problems (IVPs)

**Definition:** An initial-value problem is a problem that consists of:

- a differential equation (to be solved for an unknown function $y$)
- prescribed values for the solution and enough of it's derivatives at a particular point (the **initial point**) to determine all values for arbitrary constants in the general solution of the ODE to yield a unique **particular solution of the problem** that satisfies both the differential equation and also the additional conditions.
- **Rule:** An $n$-th order ODE needs $n$ initial values specified:

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \cdots, y^{(n)}(x_0) = y_n$$

# Example 8

Solve the first-order initial value problem

$$\begin{cases} y' = 2x - 3\sin x \\ y(0) = 0. \end{cases}$$

# Example 9

Solve the second-order initial-value problem

$$\begin{cases} y'' = \sin x \\ y(\pi) = 2 \\ y'(\pi) = -1. \end{cases}$$