3.2: Exponential Growth and Decay
and
3.3: Separable Differential Equations

Mathematics 3
Lecture 17
Dartmouth College

February 10, 2010

Exponential Growth & Decay

- **Derivatives** measure (instantaneous) rates of change.

Exponential Growth & Decay

- **Derivatives** measure (instantaneous) rates of change.

- As we have seen, a **rate of change** can be a powerful tool for expressing quantitatively a qualitative description.

Exponential Growth & Decay

- **Derivatives** measure (instantaneous) rates of change.
- As we have seen, a **rate of change** can be a powerful tool for expressing quantitatively a qualitative description.
- *"The volume is decreasing at three cubic meters per minute."*

Exponential Growth & Decay

- **Derivatives** measure (instantaneous) rates of change.

- As we have seen, a **rate of change** can be a powerful tool for expressing quantitatively a qualitative description.

- *"The volume is decreasing at three cubic meters per minute."*

  This gives an *equation* $\Longrightarrow \dfrac{dV}{dt} = -3\,\dfrac{m^3}{min}$.

Exponential Growth & Decay

- **Derivatives** measure (instantaneous) rates of change.

- As we have seen, a **rate of change** can be a powerful tool for expressing quantitatively a qualitative description.

- *"The volume is decreasing at three cubic meters per minute."*

  This gives an *equation* $\Longrightarrow \dfrac{dV}{dt} = -3\,\dfrac{m^3}{min}$.

- Suppose we know that a nation's population grows or declines depending on the birth and death rates.

Exponential Growth & Decay

- **Derivatives** measure (instantaneous) rates of change.
- As we have seen, a **rate of change** can be a powerful tool for expressing quantitatively a qualitative description.
- *"The volume is decreasing at three cubic meters per minute."*

  This gives an *equation* $\Longrightarrow \dfrac{dV}{dt} = -3\,\dfrac{m^3}{min}$.
- Suppose we know that a nation's population grows or declines depending on the birth and death rates.
- What if, at any time t, the rate of change of the size of a growing population is proportional to its size *at that moment*?

Simplest Exponential Growth & Decay Model

Let $y(t)$ be the size of the population at time t that is changing at a rate proportional to its size and let $y(0) = y_0$ be the initial size.

Simplest Exponential Growth & Decay Model

Let $y(t)$ be the size of the population at time t that is changing at a rate proportional to its size and let $y(0) = y_0$ be the initial size.

$$\text{IVP} \begin{cases} \dfrac{dy}{dt} = ky \\[2ex] y(0) = y_0 \end{cases}$$

where k is the constant of proportionality.

Simplest Exponential Growth & Decay Model

Let $y(t)$ be the size of the population at time t that is changing at a rate proportional to its size and let $y(0) = y_0$ be the initial size.

$$\text{IVP} \begin{cases} \dfrac{dy}{dt} = ky \\ \\ y(0) = y_0 \end{cases}$$

where k is the constant of proportionality.

Note that if $k > 0$, then the population is growing, and if $k < 0$, then the population is decreasing (or "decaying").

Simplest Exponential Growth & Decay Model

Let $y(t)$ be the size of the population at time t that is changing at a rate proportional to its size and let $y(0) = y_0$ be the initial size.

$$\text{IVP} \begin{cases} \dfrac{dy}{dt} = ky \\[2em] y(0) = y_0 \end{cases}$$

where k is the constant of proportionality.

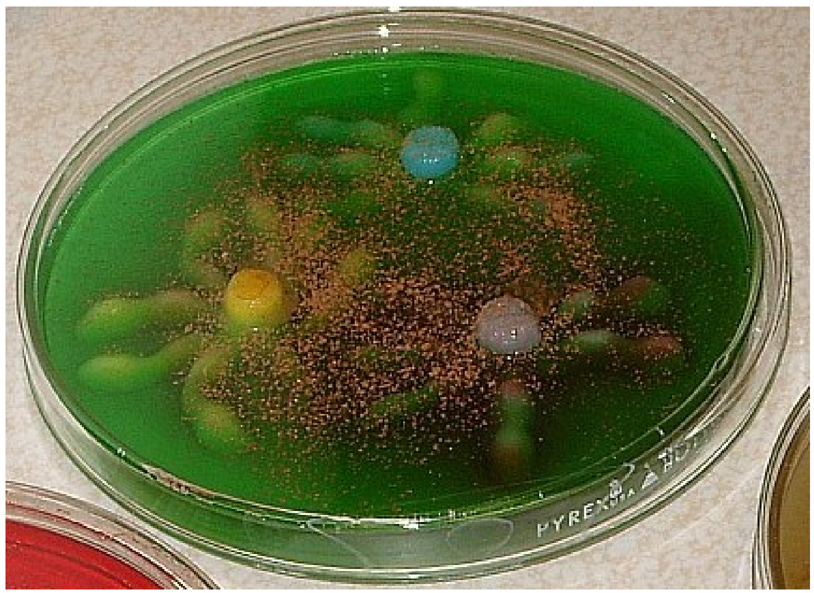
Note that if $k > 0$, then the population is growing, and if $k < 0$, then the population is decreasing (or "decaying").

Theorem. *The initial value problem above, where k is a constant, has a unique solution*

$$y = y(t) = y_0 e^{kt}.$$

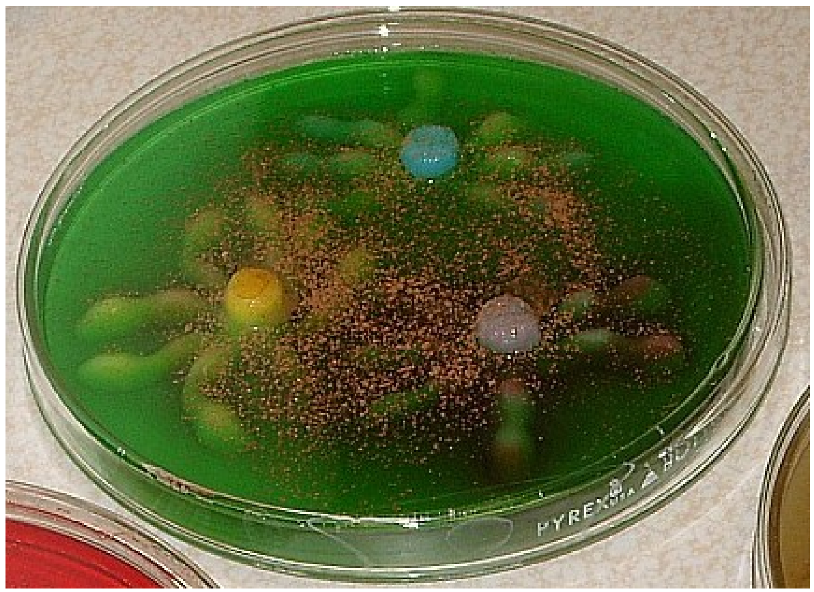
Example 1: Bacteria in a Culture

Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 7,000 cells initially and 9,000 after 12 hours, how many will be present after 36 hours?



Example 1: Bacteria in a Culture

Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 7,000 cells initially and 9,000 after 12 hours, how many will be present after 36 hours?



Answer: There are $\approx$ 14,880 cells in the culture after 36 hours.

Doubling Time and Half-Life

- In an exponential growth model $(k > 0)$, the doubling time is the length of time required for the population to double.

Doubling Time and Half-Life

- In an exponential growth model $(k > 0)$, the doubling time is the length of time required for the population to double.

- In a decay model $(k < 0)$, the half-life is the length of time required for the population to be reduced to half its size. (See half-life applet.) This is useful in studying **radioactive** elements.

Doubling Time and Half-Life

- In an exponential growth model ($k > 0$), the doubling time is the length of time required for the population to double.

- In a decay model ($k < 0$), the half-life is the length of time required for the population to be reduced to half its size. (See half-life applet.) This is useful in studying **radioactive** elements.

- A characteristic of exponential models is that these numbers are **independent** of the point in time from which the measurement begins!

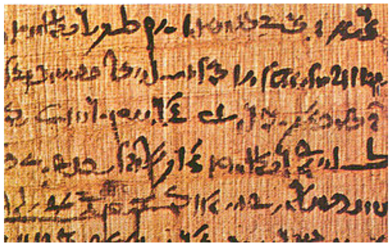
Example 2: Radioactive Decay

Carbon-14 ^{14}C is a radioactive isotope of carbon that has a half-life of $\approx$ 5,730 years, which makes it highly useful in **radiocarbon dating** of ancient artifacts and remains that contain plant/animal residue.

Example 2: Radioactive Decay

Carbon-14 ^{14}C is a radioactive isotope of carbon that has a half-life of $\approx$ 5,730 years, which makes it highly useful in **radiocarbon dating** of ancient artifacts and remains that contain plant/animal residue.

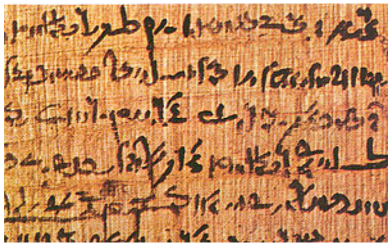
Suppose an Egyptian papyrus parchment has 66.77% as much ^{14}C as does similar papyrus plant material on Earth today. Estimate the age of the parchment and which Pharaoh it was produced under. Use the Pharaonic timeline listed at this website.



Example 2: Radioactive Decay

Carbon-14 ^{14}C is a radioactive isotope of carbon that has a half-life of $\approx$ 5,730 years, which makes it highly useful in **radiocarbon dating** of ancient artifacts and remains that contain plant/animal residue.

Suppose an Egyptian papyrus parchment has 66.77% as much ^{14}C as does similar papyrus plant material on Earth today. Estimate the age of the parchment and which Pharaoh it was produced under. Use the Pharaonic timeline listed at this website.



Solution: The parchment is $\approx$ 3,339 years old (ca. 1329 B.C.) which places it in the reign of Pharaoh Tutankhamun (1334 -1325 B.C.) of the 18th Dynasty.

Newton's Law of Cooling 

A hot/warm object introduced into a medium (e.g., room, fridge, water) maintained at a *fixed* cooler temperature will cool at a rate *proportional* to the **difference** between its own temperature and that of the surrounding medium.

Newton's Law of Cooling 

A hot/warm object introduced into a medium (e.g., room, fridge, water) maintained at a *fixed* cooler temperature will cool at a rate *proportional* to the **difference** between its own temperature and that of the surrounding medium.

If $y(t)$ is the temp at time t of an object (with initial temp y_0) placed into a medium of *fixed* temp $T_m < y_0$ then we have the IVP:

Newton's Law of Cooling 

A hot/warm object introduced into a medium (e.g., room, fridge, water) maintained at a *fixed* cooler temperature will cool at a rate *proportional* to the **difference** between its own temperature and that of the surrounding medium.

If $y(t)$ is the temp at time t of an object (with initial temp y_0) placed into a medium of *fixed* temp $T_m < y_0$ then we have the IVP:

$$\begin{cases} \dfrac{dy}{dt} = k(y - T_m) \\ \\ y(0) = y_0 \end{cases}$$

Newton's Law of Cooling 

A hot/warm object introduced into a medium (e.g., room, fridge, water) maintained at a *fixed* cooler temperature will cool at a rate *proportional* to the **difference** between its own temperature and that of the surrounding medium.

If $y(t)$ is the temp at time t of an object (with initial temp y_0) placed into a medium of *fixed* temp $T_m < y_0$ then we have the IVP:

$$\begin{cases} \dfrac{dy}{dt} = k(y - T_m) \\ \\ y(0) = y_0 \end{cases}$$

Solution: $y - T_m = (y_0 - T_m)e^{kt} \Rightarrow y = T_m + (y_0 - T_m)e^{kt}$.

Newton's Law of Cooling 

A hot/warm object introduced into a medium (e.g., room, fridge, water) maintained at a *fixed* cooler temperature will cool at a rate *proportional* to the **difference** between its own temperature and that of the surrounding medium.

If $y(t)$ is the temp at time t of an object (with initial temp y_0) placed into a medium of *fixed* temp $T_m < y_0$ then we have the IVP:

$$\begin{cases} \dfrac{dy}{dt} = k(y - T_m) \\ \\ y(0) = y_0 \end{cases}$$

Solution: $y - T_m = (y_0 - T_m)e^{kt} \Rightarrow y = T_m + (y_0 - T_m)e^{kt}$.

NB: This also describes how cool objects warm up if $T_m > y_0$...

Forensics and Differential Equations

Newton's Law of Cooling can be used in forensics to estimate the time of death, if the victim is found before reaching room temperature in a room of constant temperature. If the temperature is in degrees Fahrenheit, then $k \approx -0.05$.

Forensics and Differential Equations

Newton's Law of Cooling can be used in forensics to estimate the time of death, if the victim is found before reaching room temperature in a room of constant temperature. If the temperature is in degrees Fahrenheit, then $k \approx -0.05$.

$$\begin{cases} \dfrac{dy}{dt} = -0.05(y - T_m) \\ \\ y(0) = 98.6°F \end{cases}$$

where T_m = (fixed) temperature of the crime scene room/area.

Example 3 (CSI: Vermont)

Detective Dan Whit found the stiff at 11 am on New Year's Day in the Norwich Inn room with a deadly kitchen knife wound. He immediately measured the room and the body and found them to be $65°$F and $80°$F, respectively. The cook was implicated in the murder but she left work early at 3:45 pm and was in Boston by 6:23 pm. Could she be guilty?

Example 3 (CSI: Vermont)

Detective Dan Whit found the stiff at 11 am on New Year's Day in the Norwich Inn room with a deadly kitchen knife wound. He immediately measured the room and the body and found them to be $65°$F and $80°$F, respectively. The cook was implicated in the murder but she left work early at 3:45 pm and was in Boston by 6:23 pm. Could she be guilty?

$$\begin{cases} \dfrac{dy}{dt} = -0.05(y - 65) \\ \\ y(0) = 98.6°F \end{cases}$$

Example 3 (CSI: Vermont)

Detective Dan Whit found the stiff at 11 am on New Year's Day in the Norwich Inn room with a deadly kitchen knife wound. He immediately measured the room and the body and found them to be $65°$F and $80°$F, respectively. The cook was implicated in the murder but she left work early at 3:45 pm and was in Boston by 6:23 pm. Could she be guilty?

$$\begin{cases} \dfrac{dy}{dt} = -0.05(y - 65) \\ \\ y(0) = 98.6°F \end{cases}$$

Solution: The victim died ≈ 16 hours before 11 am, i.e., around 7:00 pm on New Year's Eve. Thus, the cook is innocent!

Separable Differential Equations

- A first-order differential equation in x and y is called separable if it is of the form
$$\frac{dy}{dx} = g(x)h(y).$$

- That is the x's and dx's can be put on one side of the equation and the y's and dy's on the other (i.e., they "separate out")

$$\frac{1}{h(y)}\, dy = g(x)\, dx$$

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

Example 4 (Compare Ex 3(c) from Monday)

- Solve the IVP

$$\begin{cases} \dfrac{dy}{dx} = x^2 y^3 \\ \\ y(3) = 1 \end{cases}$$

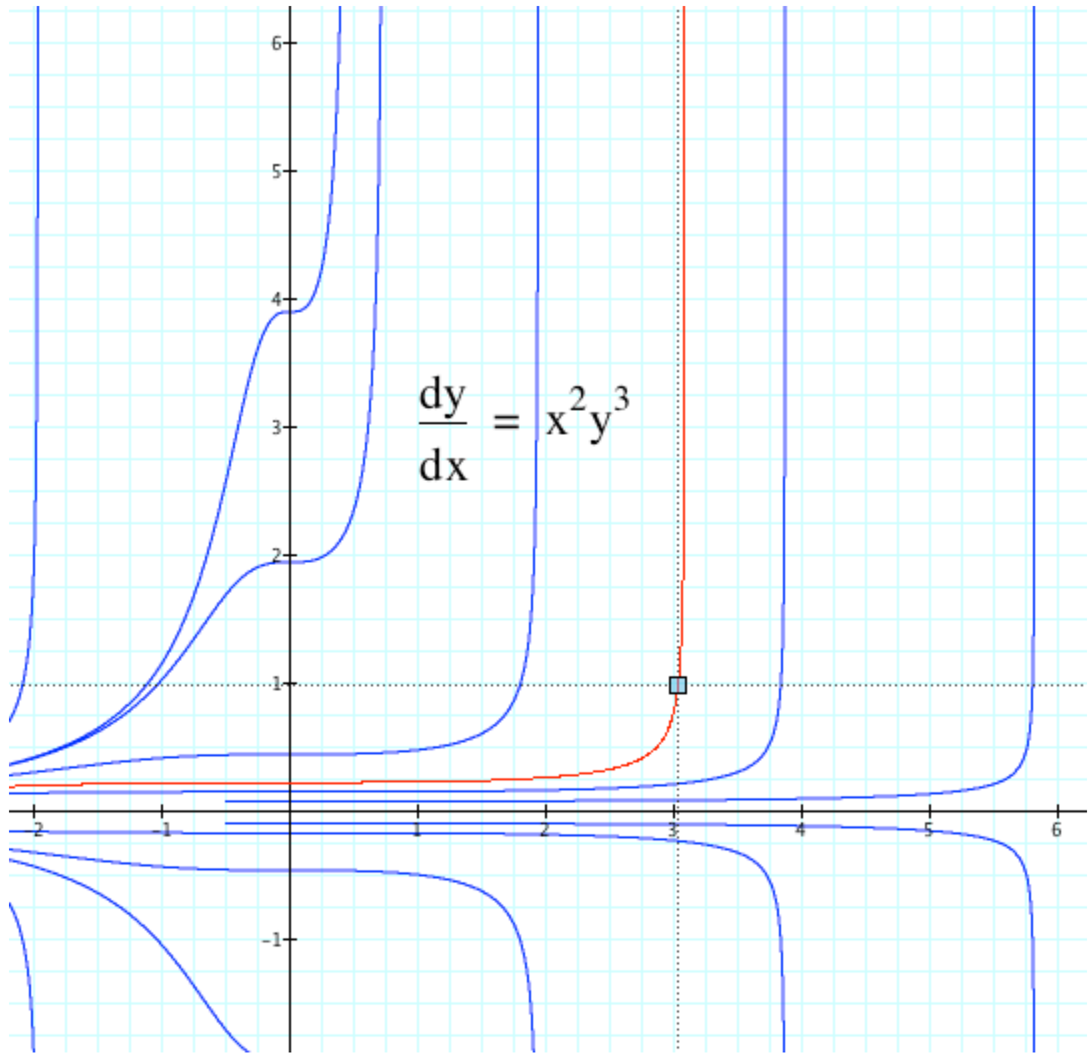
Example 4 (Compare Ex 3(c) from Monday)

- Solve the IVP

$$\begin{cases} \dfrac{dy}{dx} = x^2 y^3 \\ \\ y(3) = 1 \end{cases}$$

- The general solution is

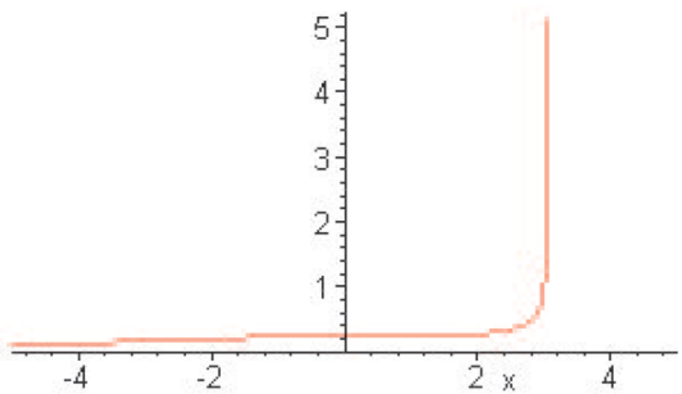
$$-\frac{1}{2y^2} = \frac{x^3}{3} + C.$$


$\frac{dy}{dx} = x^2y^3$
6
5
4
3
2
1
-1
-2
-1
1
2
3
4
5
6

- From $y(3) = 1$, we find the particular solution:

- From $y(3) = 1$, we find the particular solution:

$$
\begin{aligned}
C &= -\frac{19}{2} \\
y &= \sqrt{\frac{1}{19 - \frac{2x^3}{3}}}
\end{aligned}
$$



Justification for the Method of Separation of Variables

We need to show that given the equation

$$\frac{dy}{dx} = g(x)h(y) \Rightarrow \int \frac{1}{h(y)}\, dy = \int g(x)\, dx\,???$$

i.e., does the antiderivative of $\frac{1}{h(y)}$ as a function of y equal the antiderivative of $g(x)$ as a function of x? Let $y = f(x)$ be any solution:

$$\begin{aligned} y' &= h(y)g(x) \\ f'(x) &= h(f(x))g(x) \\ \frac{f'(x)}{h(f(x))} &= g(x) \end{aligned}$$

Let $H(y)$ be any antiderivative of $1/h(y)$

$$\begin{aligned}\frac{d}{dx}H(f(x)) &= H'(f(x))f'(x) \\ &= f'(x)\frac{1}{h(f(x))} \\ &= g(x)\end{aligned}$$

Let $H(y)$ be any antiderivative of $1/h(y)$

$$\begin{aligned} \frac{d}{dx} H(f(x)) &= H'(f(x)) f'(x) \\ &= f'(x) \frac{1}{h(f(x))} \\ &= g(x) \end{aligned}$$

So $H(f(x))$ is an antiderivative of $g(x)$....

Thus, the solution $y = f(x)$ indeed satisfies the equation

$$\int \frac{1}{h(y)} dy = H(y) = H(f(x)) = \int g(x) \mathrm{d}x \qquad ☺$$

Example 5

- Solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

Example 5

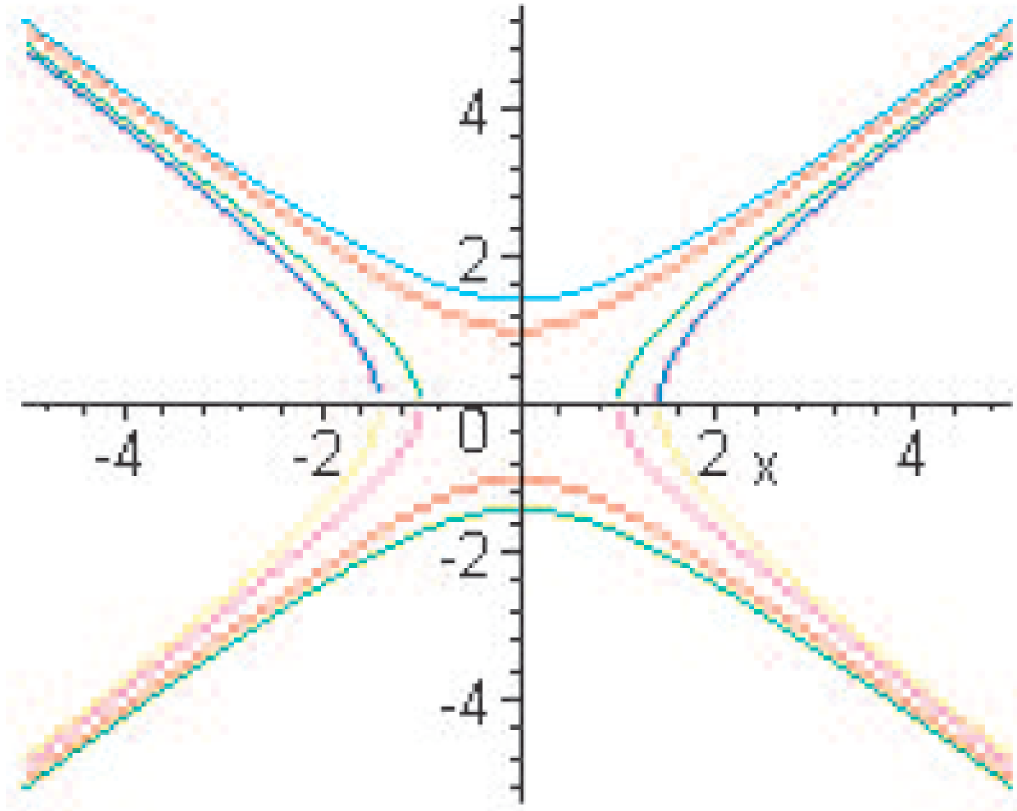
- Solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

- Solutions are of the form

$$y^2 - x^2 = C$$

which represent hyperbolae in the xy-plane.



4
2
-4
-2
0
2
x
4
-2
-4

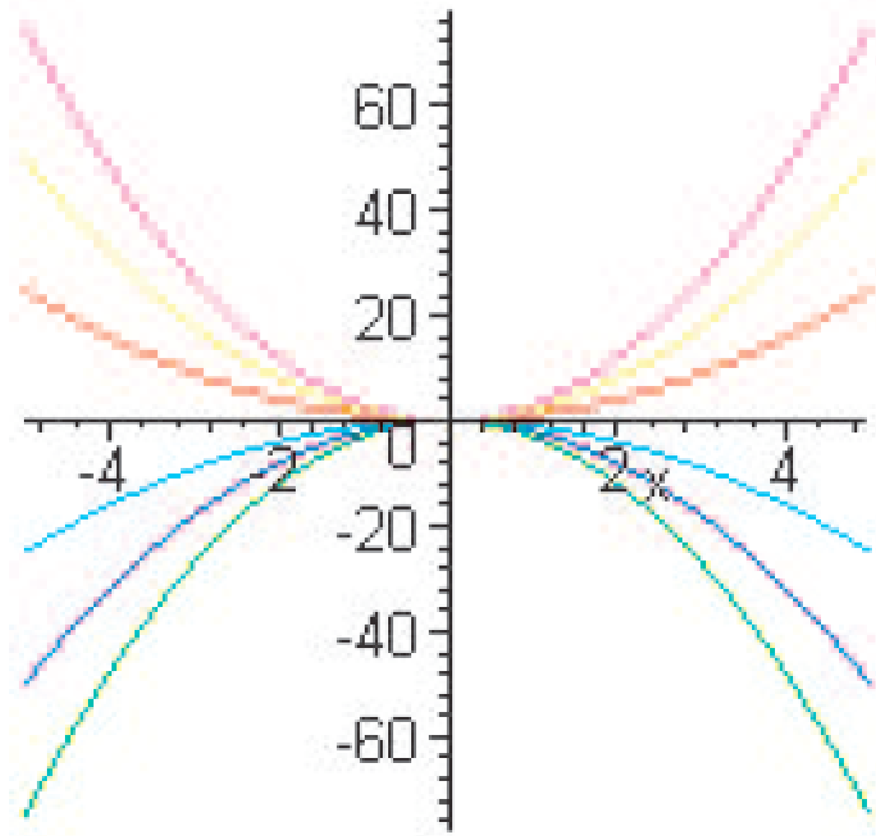
Example 6

- Solve
$$\frac{dy}{dx} = \frac{2y}{x}.$$

Example 6

- Solve
$$\frac{dy}{dx} = \frac{2y}{x}.$$

- The general solution is
$$y = Cx^2$$
which represent parabolae in the xy-plane.



60
40
20
0
-20
-40
-60
-4
-2
2
x
4

Example 7

- Solve

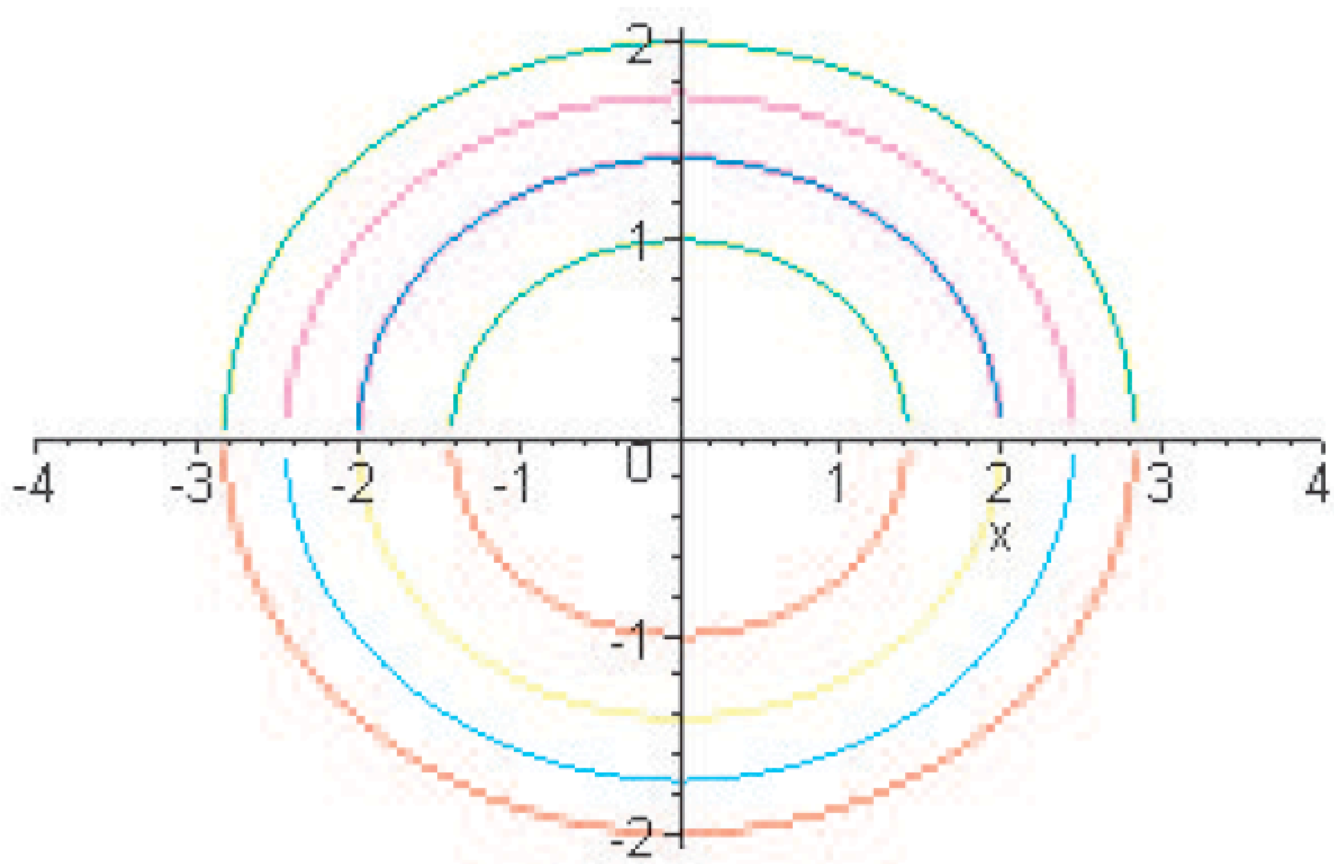
$$\frac{dy}{dx} = -\frac{x}{2y}.$$

Example 7

- Solve
$$\frac{dy}{dx} = -\frac{x}{2y}.$$

- The general solution is
$$2y^2 + x^2 = C.$$

which represent ellipses in the xy-plane.


2
1
-1
-2
-4
-3
-2
-1
0
1
2
3
4
x

Example 8

Use separation of variables to solve the IVP

$$\begin{cases} x^3 + (y+1)^2 \dfrac{dy}{dx} = 0 \\ \\ y(0) = 1 \end{cases}$$

Example 8

Use separation of variables to solve the IVP

$$\begin{cases} x^3 + (y+1)^2 \dfrac{dy}{dx} = 0 \\ \\ y(0) = 1 \end{cases}$$

General Solution: $3x^4 + 4(y+1)^3 = C$

Example 8

Use separation of variables to solve the IVP

$$\begin{cases} x^3 + (y+1)^2 \dfrac{dy}{dx} = 0 \\ \\ y(0) = 1 \end{cases}$$

General Solution: $3x^4 + 4(y+1)^3 = C$

When $x = 0$, $y = 1 \Longrightarrow C = 3 \cdot 0 + 4(2)^3 = 32$.

Particular Solution: $3x^3 + 4(y+1)^3 = 32$

Example 8

Use separation of variables to solve the IVP

$$\begin{cases} x^3 + (y+1)^2 \dfrac{dy}{dx} = 0 \\ \\ y(0) = 1 \end{cases}$$

General Solution: $3x^4 + 4(y+1)^3 = C$

When $x = 0$, $y = 1 \implies C = 3 \cdot 0 + 4(2)^3 = 32$.

Particular Solution: $3x^3 + 4(y+1)^3 = 32$

$$\implies y = -1 + \sqrt[3]{8 - \tfrac{3}{4}x^4} \quad \text{(Check on Graph Calc!)}$$