3.5: Euler's Method (cont'd)
and
3.7: Population Modeling

Mathematics 3
Lecture 19
Dartmouth College

February 15, 2010

– Typeset by FoilTEX –

Example 1

Let's consider a very simple first-order IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

Example 1

Let's consider a very simple first-order IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

This is clearly separable, so we try to solve as follows:

$$dy = e^{-x^2}\,dx \Longrightarrow y = \int e^{-x^2}\,dx = ???$$

Example 1

Let's consider a very simple first-order IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

This is clearly separable, so we try to solve as follows:

$$dy = e^{-x^2}\,dx \Longrightarrow y = \int e^{-x^2}\,dx = ???$$

Problem: There is NO known elementary formula for this integral!!!

Recall: Euler's Method Algorithm

To calculate a numerical approximate value to the solution value $y(b)$ of

$$\text{IVP}\begin{cases} \dfrac{dy}{dx} = F(x,y) \\ \\ y(x_0) = y_0 \end{cases}$$

on the interval $[x_0, b]$ (where $x_0 < b$) using $N \geq 1$ steps:

1.) Use the increment $h = \Delta x = (b - x_0)/N$.
2.) Start with the initial (value) point $P_0 = (x_0, y_0)$. Set $n = 0$.
3.) Given the "old point" (x_n, y_n), the "next point" (x_{n+1}, y_{n+1}) of the approximate solution is:

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + hF(x_n, y_n) \end{cases}$$

4.) If $n + 1 = N$, $x_N = b$, STOP. Use $y_N \approx y(b)$. Otherwise increase n by 1 and GOTO Step 3.

Recall: Euler's Method Algorithm

To calculate a numerical approximate value to the solution value $y(b)$ of

$$\text{IVP}\begin{cases}\dfrac{dy}{dx} = F(x,y)\\[2em] y(x_0) = y_0\end{cases}$$

on the interval $[x_0, b]$ (where $x_0 < b$) using $N \geq 1$ steps:

1.) Use the increment $h = \Delta x = (b - x_0)/N$.
2.) Start with the initial (value) point $P_0 = (x_0, y_0)$. Set $n = 0$.
3.) Given the "old point" (x_n, y_n), the "next point" (x_{n+1}, y_{n+1}) of the approximate solution is:

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + hF(x_n, y_n)\end{cases}$$

4.) If $n + 1 = N$, $x_N = b$, STOP. Use $y_N \approx y(b)$. Otherwise increase n by 1 and GOTO Step 3.

NOTE: This algorithm also works if $b < x_0$, we just change $+$ in Step 3 to $-$.

Example 1 (cont'd)

Let's consider that unsolvable IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

Example 1 (cont'd)

Let's consider that unsolvable IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

a.) Use Euler's method with $N = 2$ steps to approximate $y(8)$.

Example 1 (cont'd)

Let's consider that unsolvable IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

a.) Use Euler's method with $N = 2$ steps to approximate $y(8)$.

Answer: $y(8) \approx y_2 = 5$ but actually $y(8) = 1.88623...$ ☹

Example 1 (cont'd)

Let's consider that unsolvable IVP:

$$\begin{cases} \dfrac{dy}{dx} = e^{-x^2} \\ \\ y(0) = 1 \end{cases}$$

a.) Use Euler's method with $N = 2$ steps to approximate $y(8)$.

Answer: $y(8) \approx y_2 = 5$ but actually $y(8) = 1.88623...$ ☹

b.) Use the Euler Method applet with the steps in the table below to approximate $y(8)$.

Example 1 (cont'd)

b.) Use the Euler Method applet with the steps in the table below to approximate $y(8)$.

N	$h = \Delta x$	y_N
2	4	
4	2	
8	1	
16	0.5	
32	0.25	
8,000	0.001	

Example 1 (cont'd)

b.) Use the Euler Method applet with the steps in the table below to approximate $y(8)$.

N	$h = \Delta x$	y_N
2	4	5
4	2	3.03663
8	1	2.38632
16	0.5	2.13623
32	0.25	2.01123
8,000	0.001	1.88673

Recall that $y(8) = 1.88623...$

Example 1 (cont'd)

b.) Use the Euler Method applet with the steps in the table below to approximate $y(8)$.

N	$h = \Delta x$	y_N
2	4	5
4	2	3.03663
8	1	2.38632
16	0.5	2.13623
32	0.25	2.01123
8,000	0.001	1.88673

Recall that $y(8) = 1.88623...$

c.) Use the applet with $h = 0.0001$ to approximate $y(-4)$.

Example 1 (cont'd)

b.) Use the Euler Method applet with the steps in the table below to approximate $y(8)$.

N	$h = \Delta x$	y_N
2	4	5
4	2	3.03663
8	1	2.38632
16	0.5	2.13623
32	0.25	2.01123
8,000	0.001	1.88673

Recall that $y(8) = 1.88623...$

c.) Use the applet with $h = 0.0001$ to approximate $y(-4)$.

Ans: $y(-4) \approx 0.11372$ which is good since $y(-4) = 0.113773...$

Example 2

Consider the (nonseparable) IVP:

$$\begin{cases} \dfrac{dy}{dx} = x\sin(xy) \\ \\ y(-4) = 5 \end{cases}$$

Use the Euler Method applet to find the increment size $h = \Delta x$ that will give an approximation to the actual solution value

$$y(9) = 2.446829...$$

accurate to 4 decimal places.

Example 2

Consider the (nonseparable) IVP:

$$\begin{cases} \dfrac{dy}{dx} = x\sin(xy) \\ \\ y(-4) = 5 \end{cases}$$

Use the Euler Method applet to find the increment size $h = \Delta x$ that will give an approximation to the actual solution value

$$y(9) = 2.446829...$$

accurate to 4 decimal places.

Answer: When $h = 0.01$, $y(9) \approx 2.44683$ accurate to 4 decimals.

How to Build (Better) Mathematical Models

1. Translate real-world problems into mathematical models.

How to Build (Better) Mathematical Models

1. Translate real-world problems into mathematical models.

2. Subject the models to mathematical analysis and prediction.

How to Build (Better) Mathematical Models

1. Translate real-world problems into mathematical models.

2. Subject the models to mathematical analysis and prediction.

3. Draw conclusions from the models.

How to Build (Better) Mathematical Models

1. Translate real-world problems into mathematical models.
2. Subject the models to mathematical analysis and prediction.
3. Draw conclusions from the models.
4. Test the conclusions in the laboratory and compare the results with the original real-world data.

How to Build (Better) Mathematical Models

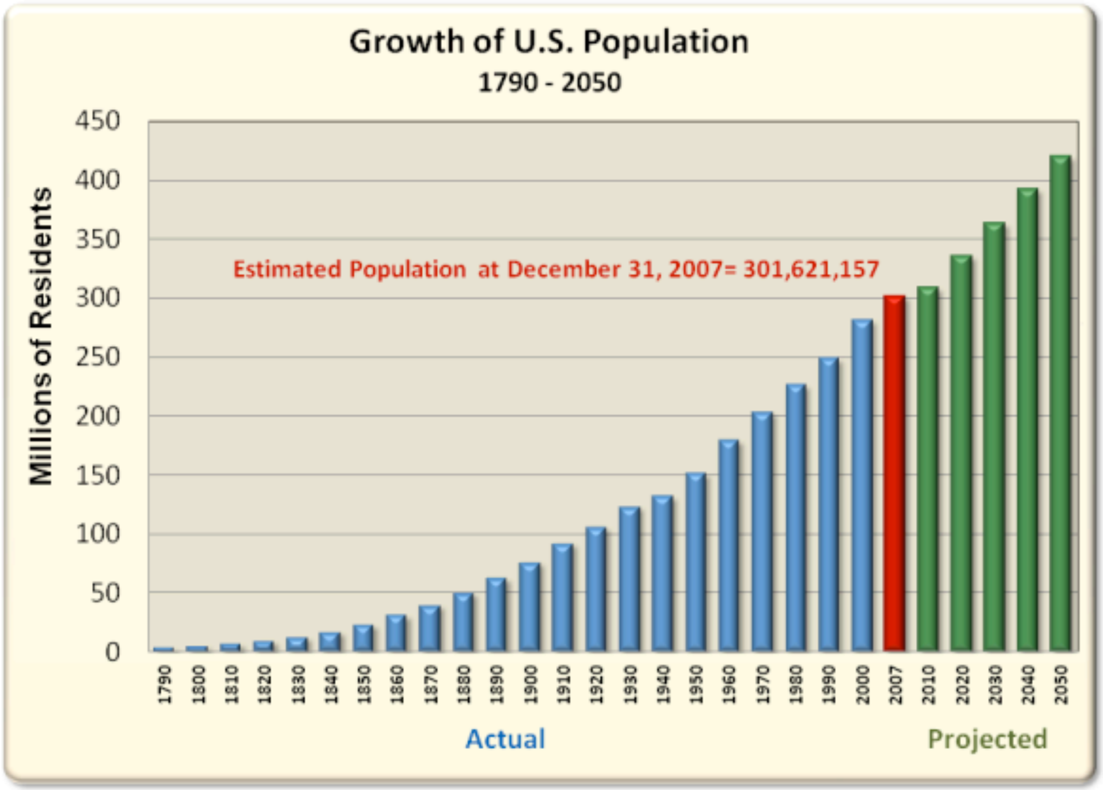
1. Translate real-world problems into mathematical models.
2. Subject the models to mathematical analysis and prediction.
3. Draw conclusions from the models.
4. Test the conclusions in the laboratory and compare the results with the original real-world data.
5. Revise the model as necessary and repeat the above steps until the model is a reliable predictor of real-world observations.

3.7: Case Study: Population Modeling

Objective: Predict the size of the population of the United States well into the 21st century.

3.7: Case Study: Population Modeling

Objective: Predict the size of the population of the United States well into the 21st century.



Source: United States Census Bureau, 2008 Statistical Abstract
(1) Publication PHC-3-1 [Table B], (2) U.S. Interim Projections by Age, Sex, Race, and Hispanic Origin [2004]

Population Census Data of the United States

The U.S. Census Bureau is required by the U. S. Constitution to do a national census of the population every 10 years.

Population Census Data of the United States

The U.S. Census Bureau is required by the U. S. Constitution to do a national census of the population every 10 years.

We will use only the populations recorded in the census of the years 1790, 1990 and 2000:

Population Census Data of the United States

The U.S. Census Bureau is required by the U. S. Constitution to do a national census of the population every 10 years.

We will use only the populations recorded in the census of the years 1790, 1990 and 2000:

Year	Population (millions)
1790	3.9
1990	248.7
2000	281.4

The Malthus Population Model

The Malthus Population Model

- The **Malthus model** for growth of a population assumes an ideal environment.

The Malthus Population Model

- The **Malthus model** for growth of a population assumes an ideal environment.

- Resources are unlimited, disease is constrained, and individuals are happy.

The Malthus Population Model

- The **Malthus model** for growth of a population assumes an ideal environment.
- Resources are unlimited, disease is constrained, and individuals are happy.
- The population increases at a rate proportional to the number of individuals present.

Malthus Model: Exponential Growth

$Q = Q(t)$ be the population of the USA (in millions) at time t.
$t =$ number of years since the 1790 census.

Malthus Model: Exponential Growth

$Q = Q(t)$ be the population of the USA (in millions) at time t.
$t =$ number of years since the 1790 census.

Starting with a population of 3.9 million in 1790, we have the IVP:

$$\begin{cases} \dfrac{dQ}{dt} = kQ \\[2ex] Q(0) = 3.9 \end{cases}$$

Malthus Model: Exponential Growth

$Q = Q(t)$ be the population of the USA (in millions) at time t.
$t =$ number of years since the 1790 census.

Starting with a population of 3.9 million in 1790, we have the IVP:

$$\begin{cases} \dfrac{dQ}{dt} = kQ \\ \\ Q(0) = 3.9 \end{cases}$$

Example 3 Use the Malthus model to do the following:
a.) Use the Population Modeling applet to guess the right value of k to give the 1990 census figure (at $t = 200$) and check your answer algebraically.
b.) Estimate the US populations in 2000 and 2010 and analyze the results. Is the Malthus model a good predictor?

The Verhulst Population Model

The **Verhulst model** assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

The Verhulst Population Model

The **Verhulst model** assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

Takes into account the effects of a limiting environment.

The Verhulst Population Model

The **Verhulst model** assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

Takes into account the effects of a limiting environment.

It is a more realistic model.

Verhulst Model: Limited Exponential Growth

The Verhulst model assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

Verhulst Model: Limited Exponential Growth

The Verhulst model assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

It replaces the growth constant k by the algebraic expression

$$k\Big(\frac{M - Q(t)}{M}\Big)$$

Verhulst Model: Limited Exponential Growth

The Verhulst model assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

It replaces the growth constant k by the algebraic expression

$$k\left(\frac{M - Q(t)}{M}\right)$$

This leads to the differential equation

$$\frac{dQ}{dt} = k\left(\frac{M - Q}{M}\right)Q.$$

The factor $\frac{M-Q}{M}$ has a value between 0 and 1:

$$0 \leq \frac{M-Q}{M} \leq 1$$

and is sometimes called the unrealized potential for pop. growth.

The factor $\frac{M-Q}{M}$ has a value between 0 and 1:

$$0 \leq \frac{M-Q}{M} \leq 1$$

and is sometimes called the unrealized potential for pop. growth.

When Q is small $(Q \approx 0)$ we have that this factor

$$\frac{M-Q}{M} \approx 1 \implies k\Big(\frac{M-Q}{M}\Big)Q \approx kQ$$

and the growth of the population is essentially exponential.

The factor $\frac{M-Q}{M}$ has a value between 0 and 1:

$$0 \leq \frac{M-Q}{M} \leq 1$$

and is sometimes called the unrealized potential for pop. growth.

When Q is small ($Q \approx 0$) we have that this factor

$$\frac{M-Q}{M} \approx 1 \implies k\Big(\frac{M-Q}{M}\Big)Q \approx kQ$$

and the growth of the population is essentially exponential.

As Q approaches its asymptotic limiting value ($Q \nearrow M$), however, the factor

$$\frac{M-Q}{M} \searrow 0$$

is close to zero, and the population grows ever more slowly. [See GC]

Objective

- The U.S. population cannot sustain exponential growth indefinitely.
- The Malthus model gives unrealistic projections of the population over the next century.
- We would like to use the Verhulst model instead to make such projections.
- We also need to assume that $Q(0) = 3.9$ million, and $M = 750$ million, the maximum value of the population $(0 \leq Q(t) \leq M)$.

Verhulst U.S. Population Model

$$\begin{cases} \dfrac{dQ}{dt} = k\Big(\dfrac{750 - Q}{750}\Big)Q \\ \\ Q(0) = 3.9 \end{cases}$$

Verhulst U.S. Population Model

$$\begin{cases} \dfrac{dQ}{dt} = k\Big(\dfrac{750 - Q}{750}\Big)Q \\ \\ Q(0) = 3.9 \end{cases}$$

Example 4 Use the Verhulst U.S. population model to do the following:

a.) Predict with the Population Modeling applet using $k = 0.0228$ the US population in the years 1990, 2000 and analyze the results. Is this a better model than the Malthus model?

b.) Now estimate the US population in the years 2010 and 2020.