# 4.4: The Fundamental Theorem of Calculus ☺
# and
# 4.5: Techniques of Integration

Mathematics 3
Lecture 23
Dartmouth College

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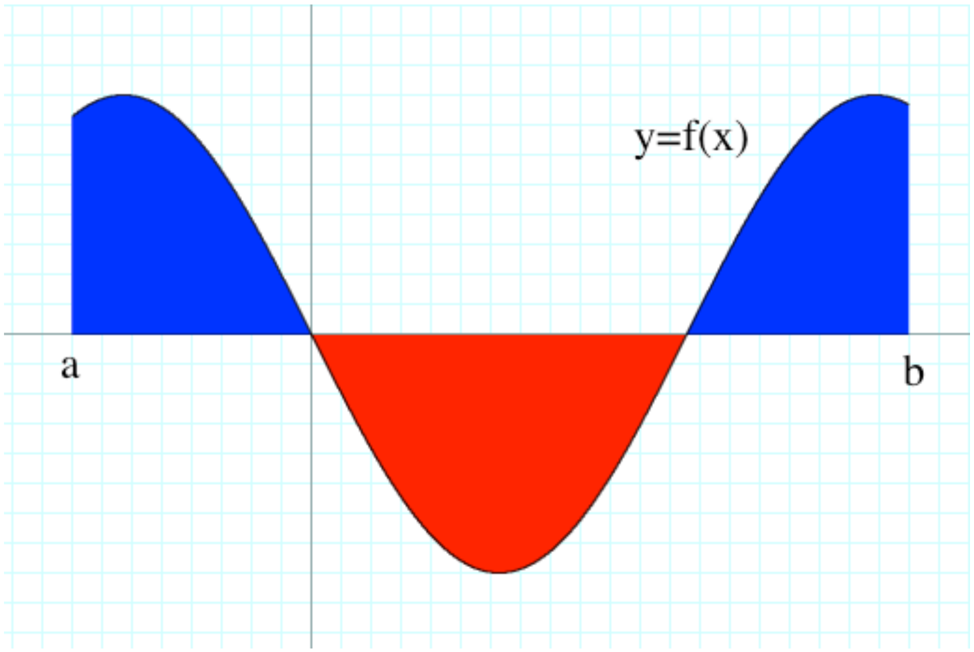
# The Definite Integral of $y = f(x)$ over $[a, b]$




$$\int_a^b f(x)\,dx = (Area\ above\ x - axis) - (Area\ below\ x - axis)$$

# The Definite Integral of $y = f(x)$ over $[a, b]$

If $f$ is a function on $[a, b]$, then we can (most simply) define the definite integral of $f$ on $[a, b]$ to be the real number which is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{(if this limit exists...)}$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$ (right endpoints). If this limit exists, the function $f$ is called Riemann integrable on $[a, b]$.

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**Example 1:** Find $f(x)$ and $a$ and $b$ so that

$$\lim_{n\to\infty} \sum_{i=1}^{n} 5\Big( -2 + \frac{3i}{n}\Big)\Big(\frac{3}{n}\Big) = \int_a^b f(x)\,dx$$

and evaluate the limit using properties of the definite integral.

# Properties of the Definite Integral

**-1.** $\displaystyle\int_a^a f(x)dx = 0$

**0.** $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$

**1.** $\displaystyle\int_a^b c\,dx = c(b-a)$

**2.** $\displaystyle\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

**3.** $\displaystyle\int_a^b c\,f(x)dx = c\int_a^b f(x)dx$

**4.** $\displaystyle\int_a^b \big(f(x)-g(x)\big)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

**5.** $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

**6.** **If** $f(x) \geq 0$ for all $a \leq x \leq b$, **then** $\displaystyle\int_a^b f(x)dx \geq 0$.

**7.** **If** $f(x) \geq g(x)$ for all $a \leq x \leq b$, **then** $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

**8.** **If** $m \leq f(x) \leq M$ for all $a \leq x \leq b$, **then** $\displaystyle m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

**NB:** This handout is posted on Blackboard in the (Documents) section.

# The Fundamental Theorem of Calculus: Part I

**Question:** We know that continuous functions are Riemann integrable (and may **not** be differentiable), but do they ever have any antiderivatives? That is, are continuous functions antidifferentiable?

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# The Fundamental Theorem of Calculus: Part I

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**Answer:** Yes, always!!

**Theorem (Part I):** Suppose that $f$ is a continuous function on the interval $I$ containing the point $a$. Define a (new) function $F$ on $I$ by the following definite integral formula:

$$F(x) = \int_a^x f(t)dt.$$

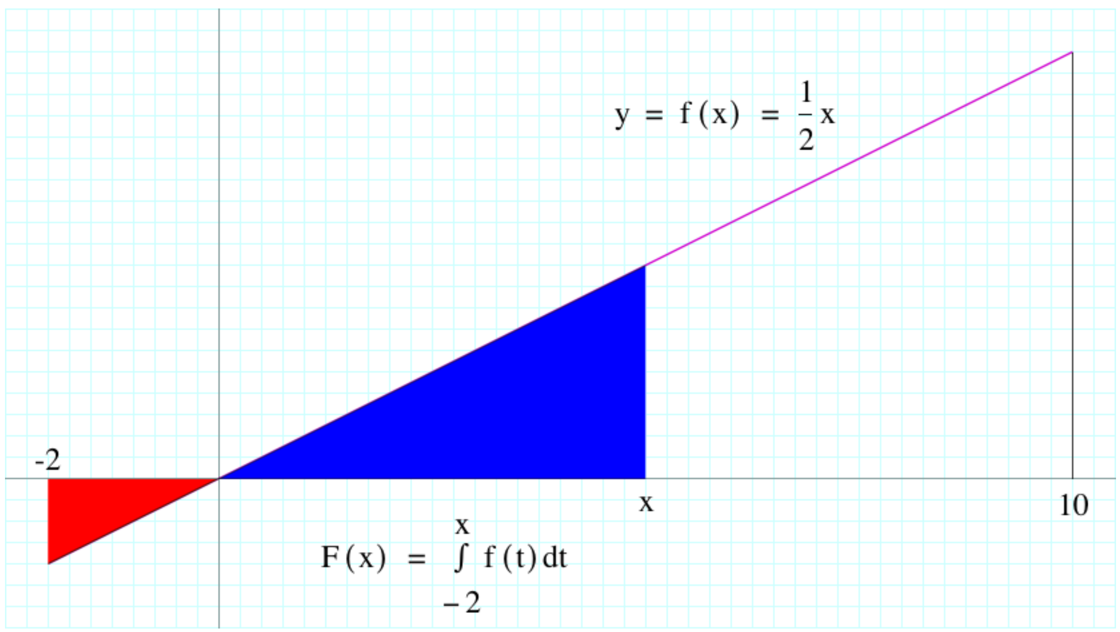
Then $F$ is differentiable on $I$ and $F'(x) = f(x)$. That is, $F$ is an antiderivative of $f$ on $I$.

# The Fundamental Theorem of Calculus: Part I

**Example 2:** Let $f(x) = \frac{1}{2}x$ on the interval $[-2, 10]$. Show that

$$F(x) = \int_{-2}^{x} f(t)\,dt = \int_{-2}^{x} 7t\,dt$$

is an antiderivative for $f$ by computing it explicitly.



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**Example 3:** Find the following derivatives.

a.) $\displaystyle \frac{d}{dx}\int_1^x t\sqrt{t^2+1}\,dt$

b.) $\displaystyle \frac{d}{dx}\int_{-\pi}^{x^2} \cos(t^5)\,dt$

c.) $\displaystyle \frac{d}{dx}\int_{x^2}^{3} e^{-t^2}\,dt$

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**Theorem (Part II):** If $G(x)$ is **ANY** antiderivative of $f$ on an interval $I$ (that is, $G'(x) = f(x)$ on $I$), then for any point $b$ in $I$,

$$\int_a^b f(x)dx = G(b) - G(a).$$

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**Notation**: $G(x)|_a^b = G(b) - G(a)$

## The Fundamental Theorem of Calculus: Part II

**Example 4:** Compute the following:

a.) $\int_0^1 (x^3 - 5x + 1)\,dx$

b.) The area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the $x$-axis, and the vertical lines $x = 0$ and $x = 2$.

c.) The average value of $y = \sec^2(x)$ on the interval $[0, \frac{\pi}{4}]$.

d.) $\int_0^{\sqrt{\ln(5\pi/6)}} \frac{d}{dt} \sin(e^{t^2})\,dt$.

# Techniques of Integration

Recall our basic (atomic) integration formulas...

$$
\begin{aligned}
\int u^r \, du &= \frac{u^{r+1}}{r+1} + C, r \neq -1 \\
\int \frac{1}{u} \, du &= \ln|u| + C \\
\int \sin u \, du &= -\cos u + C \\
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**NB:** Antidifferentiation is much harder than differentiation...

# Derivatives and Differentials

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)\,dx$$

- The last equation is the differential (form) version and we refer to the formal symbols $dy$ and $dx$ as "differentials". (Need Math 73.)

- If $y = x^3$, then $dy = 3x^2\,dx$.

- If $y = \sin 4x$, then $dy = 4\cos 4x\,dx$.

# Review: The Method of Substitution

If $u = g(x)$ is a function of $x$, and $y = f(u)$ is a function of $u$, then the Chain Rule tells us that for $y = f(u) = f(g(x))$:

$$y' = \frac{dy}{dx} = (f(g(x)))' = f'(g(x))g'(x).$$

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Substitute $u = g(x)$ and the differential $du = g'(x)\,dx$. When we make these two substitutions we get

$$\int f'(u)\,du = f(u) + C.$$

# Review: The Method of Substitution

**Example 5:** Compute the following

a.) $\displaystyle\int \frac{1}{2+3e^x}\,dx$

b.) Find the area under $y = \dfrac{x}{\sqrt{x^2-1}}$ on the interval $[2,5]$.

c.) $\displaystyle\int_e^{e^2} \frac{\ln x}{x}\,dx$

d.) $\displaystyle\int_0^{\pi/4} \tan x\,dx$