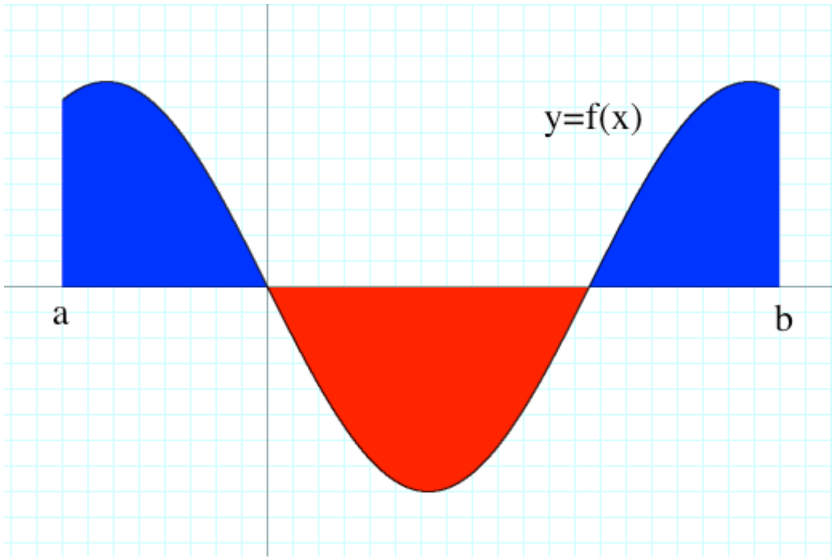
4.5: Techniques of Integration (cont'd)
and
4.7: Areas Between Curves

Mathematics 3
Lecture 24
Dartmouth College

March 01, 2010

The Fundamental Theorem of Calculus: Part II

$$\int_a^b f(x)\,dx = (Area\ above\ x-axis) - (Area\ below\ x-axis)$$



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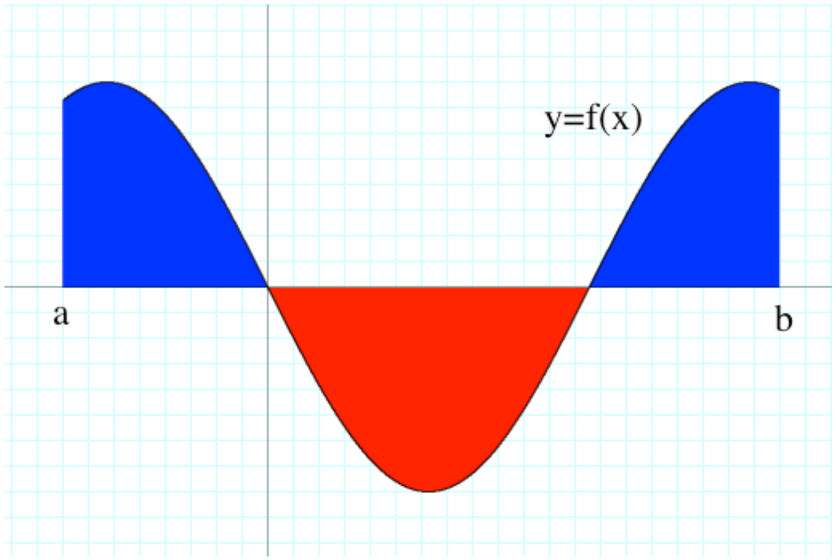

Thm: If $G(x)$ is **ANY** (known...) antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

The Atiyah-Singer Index Theorem

In my research, I use an important generalization of this theorem...

Thm: Let D be an elliptic differential operator on a smooth compact multi-dimensional manifold M ($=$ curved higher-dimensional space) then

$$\int_{T^*M} ch(\sigma(D)) \wedge Td(M) \, \mathrm{dvol}_{T^*M} = \dim \ker(D) - \dim \operatorname{coker}(D)$$

This theorem helps count the number of different solutions of certain differential equations, explains why virtual particles in quantum physics always come in particle/antiparticle pairs (e.g., electron/positron) and helps with computations in string theory!

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NOTE: This slide is **NOT** on the final exam... ☺

Technique: Integration by Parts

Besides the Method of Substitution, another technique of integration that is often useful involves a "reversing" of the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

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Integration by Parts formula: $\displaystyle\int u\,dv = uv - \int v\,du.$

Technique: Integration by Parts

Example 1 Compute the following:

$$\int x \cos(x)\, dx$$

NB: This was a problem on the midterm exam...

Technique: Integration by Parts

$$\int f(x)g'(x)\,dx = \int u\,dv = uv - \int v\,du$$

$$\begin{cases} u = f(x) & v = g(x) \\ du = f'(x)\,dx & dv = g'(x)\,dx \end{cases}$$

• Need to choose u and dv from the given integral carefully!

• Try letting dv be the most complicatd portion of the integral that fits a basic (atomic) integration formula. Then u will be the remaining stuff in the integrand.

• Or, try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining stuff in the integrand.

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Who cares? Cartoon animators do!

Technique: Integration by Parts

Example 2 Compute the following:

a.) $\int xe^x \, dx$

b.) $\int_1^e \ln x \, dx$

c.) $\int x^2 \sin x \, dx$

d.) Find the area under the graph of $y = e^{\sqrt{x}}$ over the interval $[1, 4]$.

e.) Find the general solution to the ODE $\csc(x)y' - x^2 = 0$.

Hints for Integrals using Integration by Parts

1.) For integrals of the form

$$\int x^n e^{ax}\, dx, \quad \int x^n \sin(ax)\, dx, \quad \text{or} \quad \int x^n \cos(ax)\, dx$$

let $u = x^n$ and let $dv = e^{ax}\, dx,\ \sin(ax)\, dx$, or $\cos(ax)\, dx$.

2.) For integrals of the form

$$\int e^{ax} \sin(bx)\, dx \quad \text{or} \quad \int e^{ax} \cos(bx)\, dx$$

let $u = \sin(bx)$ or $\cos(bx)$ and let $dv = e^{ax}\, dx$.

The Area Between Two Curves

- We know that if f is a continuous **nonnegative** function on the interval $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is equal to the area under the graph of f and above the interval.

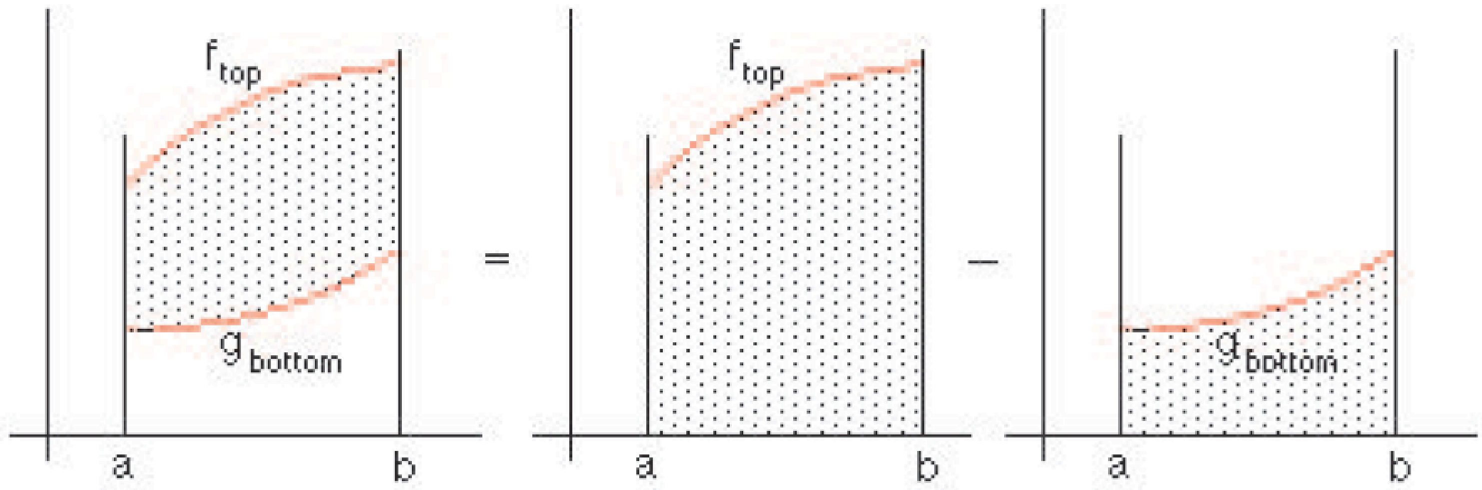
- Suppose we are given two continuous functions, f_{top} and g_{bottom} defined on the interval $[a, b]$, with

$$g_{bottom}(x) \leq f_{top}(x)$$

  for all x in the interval.

- How do we find the area bounded between the two functions over that interval?

The Area Between Two Curves



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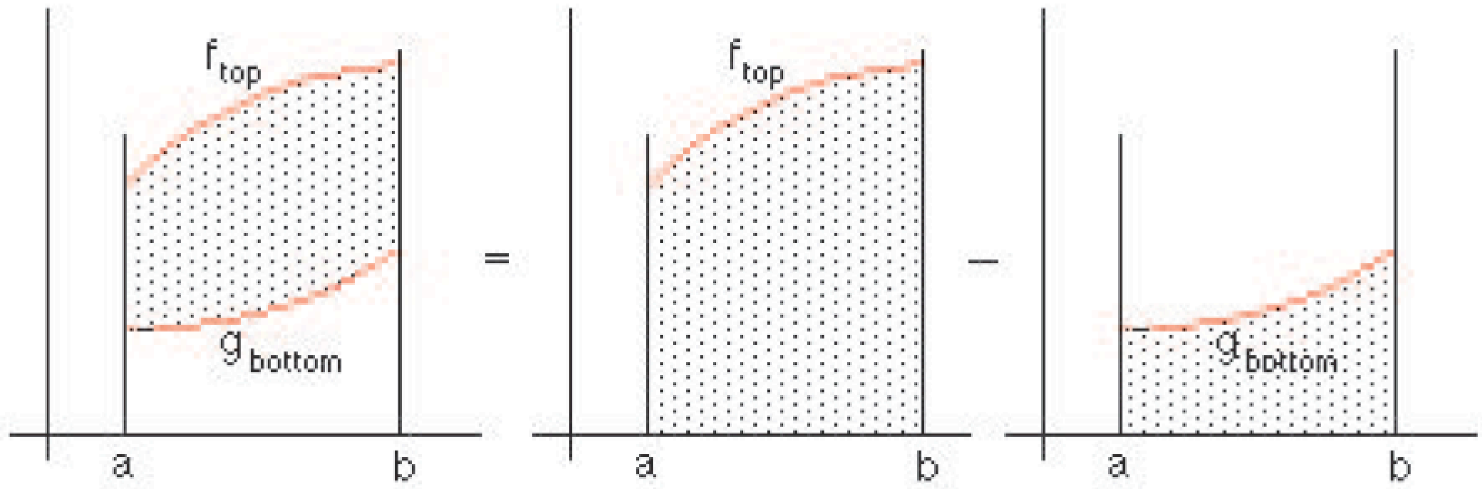

$$A = \int_a^b f_{top}(x)\,dx - \int_a^b g_{bottom}(x)\,dx = \int_a^b \big(f_{top}(x) - g_{bottom}(x)\big)\,dx$$

Example 3

Find the area of the region between the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.

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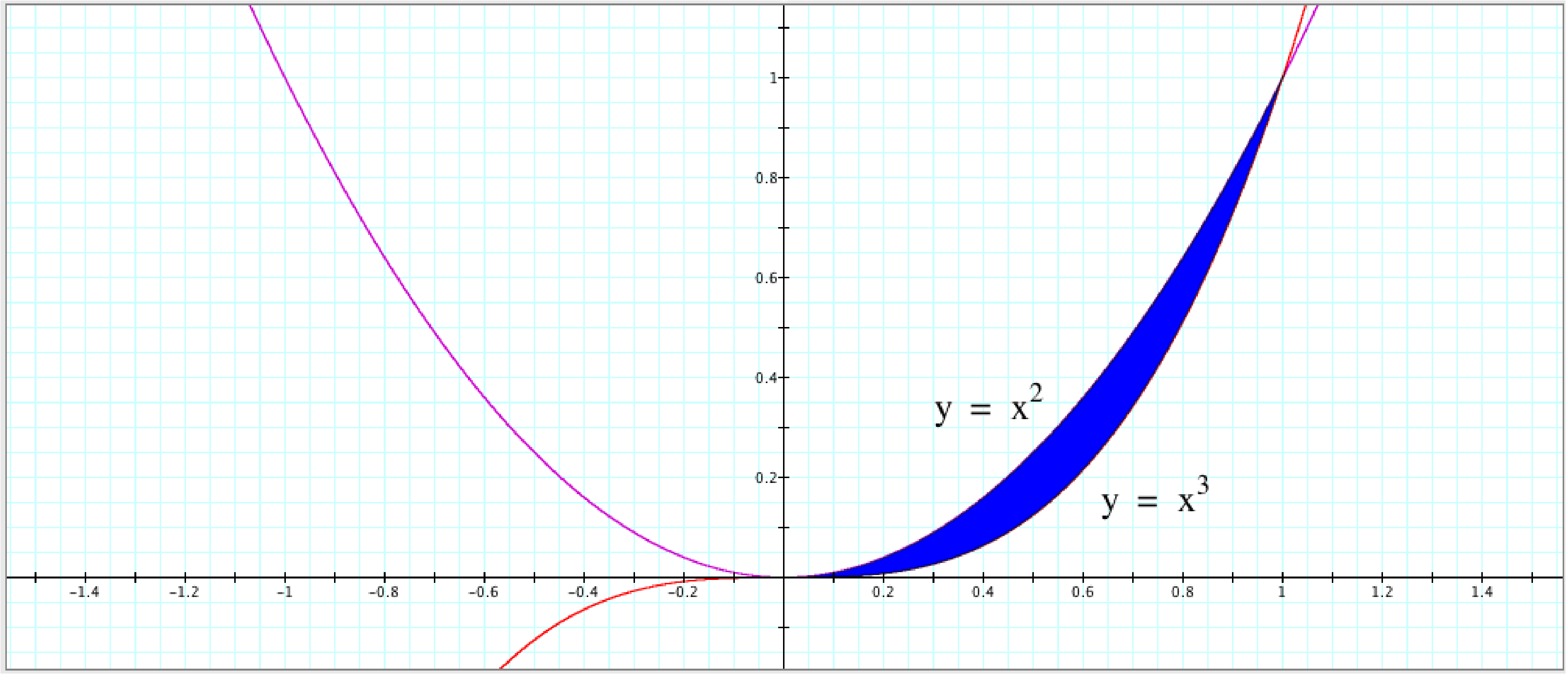

Example 4

Find the area of the region between $y = e^x$ and $y = 1/(1+x)$ on the interval $[0, 1]$.

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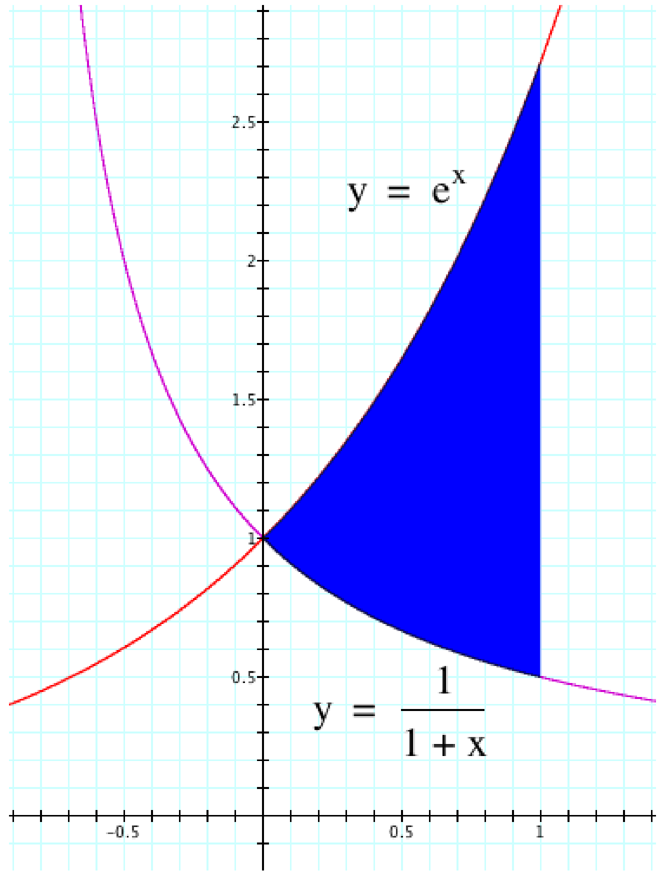

Example 5

Find the area of the region bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.

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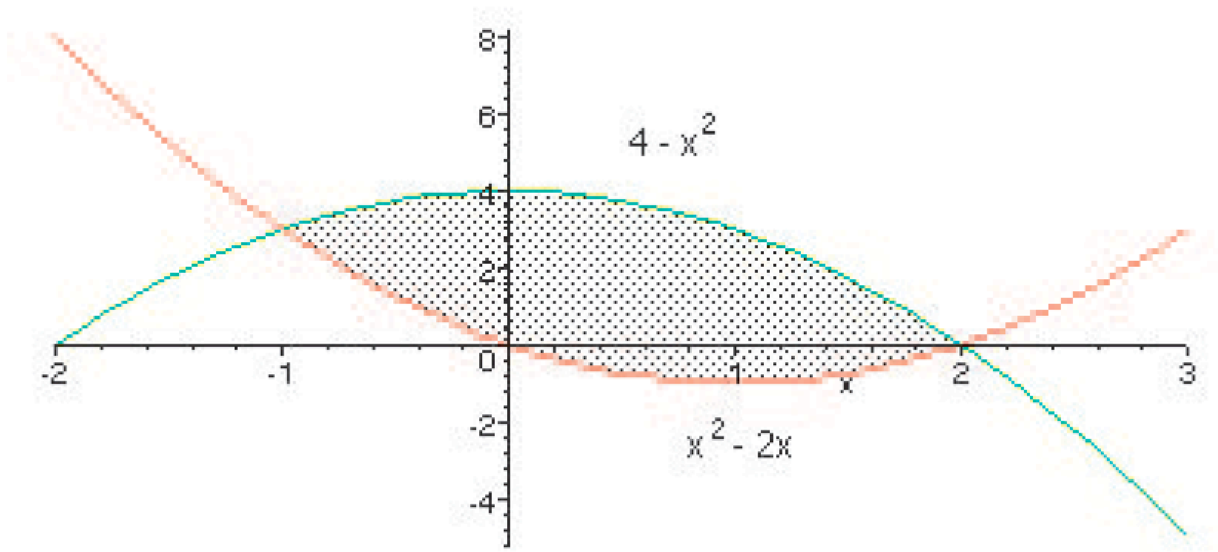

Example 6

Find the area of the region bounded by the two curves $y = x^3 - 9x$ and $y = 9 - x^2$.

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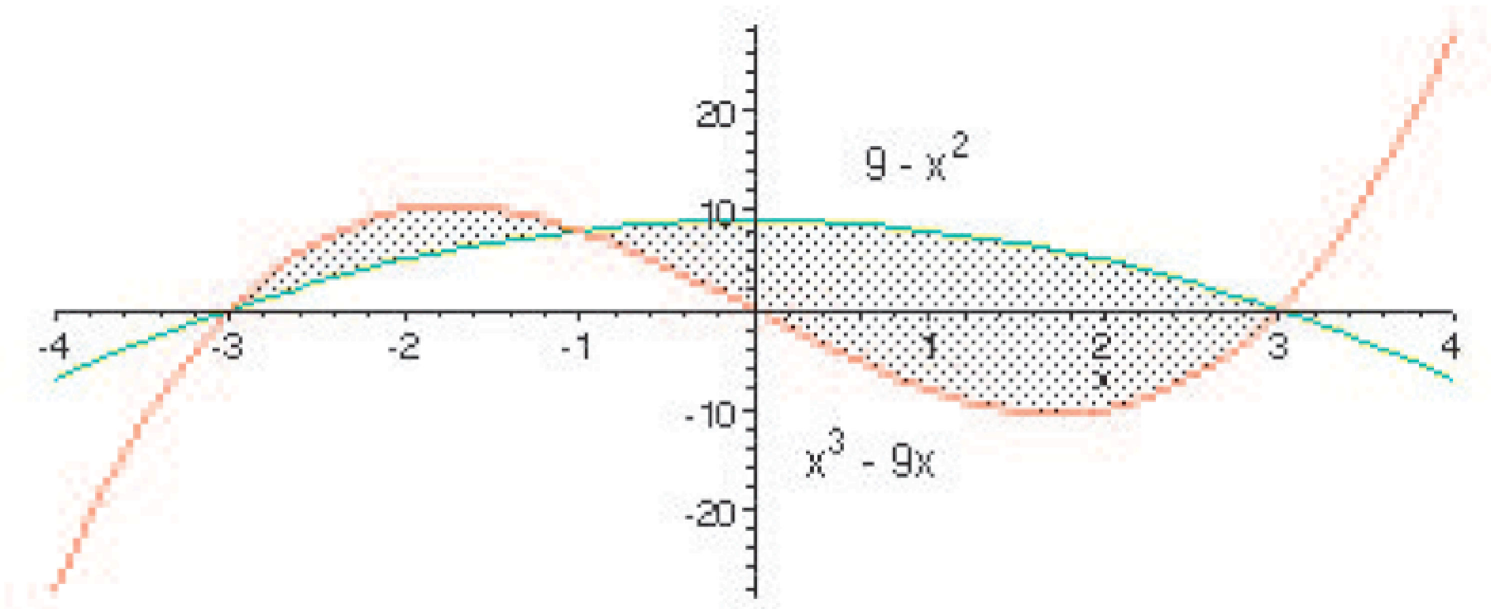

Functions of y

We could just as well consider two functions of y, say, $x = f_{Left}(y)$ and $x = g_{Right}(y)$ defined on the interval $[c, d]$.

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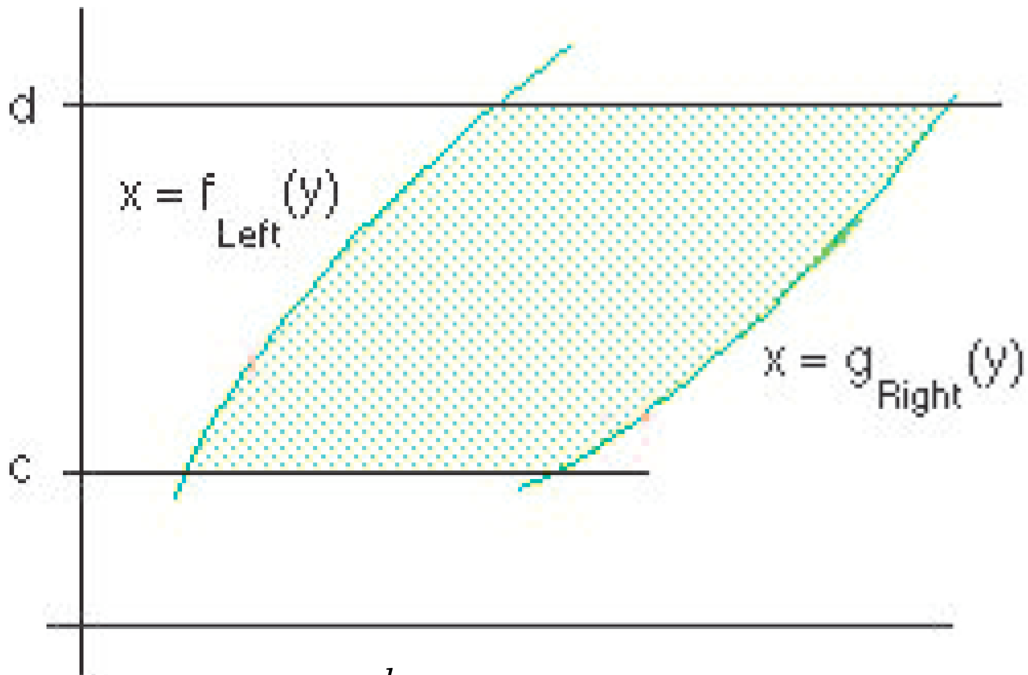

$$\text{Area } A = \int_c^d \Big(g_{Right}(y) - f_{Left}(y)\Big)\, dy$$

Example 7

Find the area bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

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